Section 6.1, 6.2, 6.3, 6.4

Exercise 1. Write the sum in sigma notation:

1. $\frac{3}{\sqrt{3}}+\frac{3}{\sqrt{4}}+\frac{3}{\sqrt{5}}+\frac{3}{\sqrt{6}}+\frac{3}{\sqrt{7}}+\frac{3}{\sqrt{8}}+\frac{3}{\sqrt{9}}+\frac{3}{\sqrt{10}}+\frac{3}{\sqrt{11}}$. $= \sum_{k=3}^{11} \frac{3}{\sqrt{k}}$

2. $n^2+(n+1)^2+(n+2)^2+(n+3)^2+(n+4)^2+(n+5)^2+(n+6)^2+(n+7)^2+(n+8)^2$. $= \sum_{i=n}^{n+8} i^2 = \sum_{j=0}^{8}(n+j)^2$

3. $\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\frac{1}{10}-\frac{1}{12}+\frac{1}{14}-\frac{1}{16}+\frac{1}{18}-\frac{1}{20}$.

$$= \frac{1}{2\times1} - \frac{1}{2\times2} + \frac{1}{2\times3} - \frac{1}{2\times4} + \frac{1}{2\times5} - \cdots - \frac{1}{2\times10} = \sum_{i=1}^{10} \frac{(-1)^i}{2i}$$

Exercise 2. Find the value of the sum

1. $\sum_{j=1}^{100} 2j+5$. $= 2\sum_{j=1}^{100} j + \sum_{j=1}^{100} 5 = 2\left(\frac{100(100+1)}{2}\right) + 5\times100 = 10600$

2. $\sum_{i=1}^{n}(3+2i)^2$. $= \sum_{i=1}^{n}(9+6i+4i^2) = \sum_{i=1}^{n} 9 + 6\sum_{i=1}^{n} i + 4\sum_{i=1}^{n} i^2$

$$= 9n + 6\,\frac{n(n+1)}{2} + 4\left(\frac{n(n+1)(2n+1)}{6}\right)$$

3. $\sum_{k=1}^{43}(-1)^k.$ $= \underbrace{\underbrace{-1}_{i=1} + \underbrace{1}_{i=2}}_{0} + \underbrace{\underbrace{-1}_{i=3} + 1}_{0} - 1 + 1 \ldots \underbrace{-1}_{i=43}$

$0 + 0 + \ldots + -1$

$= -1$

4. $\sum_{i=1}^{99}\left(\frac{1}{i}-\frac{1}{i+1}\right).$ $= \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \ldots \left(\cancel{\frac{1}{97}} - \cancel{\frac{1}{98}}\right) + \left(\cancel{\frac{1}{98}} - \cancel{\frac{1}{99}}\right) + \cancel{\frac{1}{99}} - \frac{1}{100}$

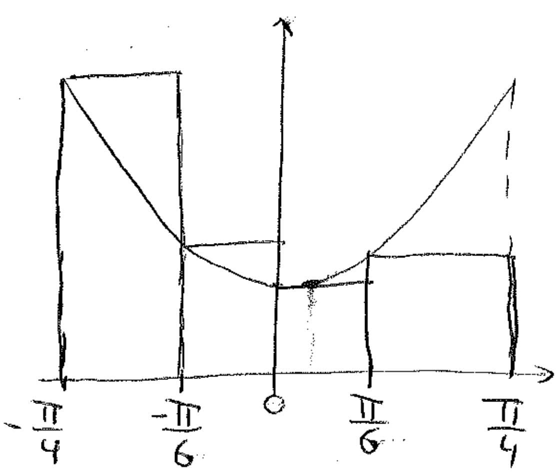
$= 1 - \frac{1}{100} = 0.99$

5. $\sum_{k=1}^{n}(i^5-(i-1)^5).$ $= (\cancel{1^5} - 0^5) + (\cancel{2^5} - \cancel{1^5}) + (\cancel{3^5} - \cancel{2^5}) + (\cancel{4^5} - \cancel{3^5}) \ldots n^5 - \cancel{(n-1)^5}$

$= -0^5 + n^5 = n^5$

Exercise 3. Given the function $f(x) = 1+\tan^2 x$ on the interval $[-\pi/4, \pi/4]$, the partition points $\{-\pi/4, -\pi/6, 0\pi/6, \pi/4\}$ and x_i^*=left endpoint,

1. Sketch the graph of f and the approximating rectangles.



2. Find $||P||$.

width of the rectangles : $-\frac{\pi}{6}+\frac{\pi}{4} = \frac{(-2+3)\pi}{12} = \frac{\pi}{12}, \frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{12}$

$$\boxed{||P|| : \frac{\pi}{6}}$$

3. Find the sum of the area of the approximating rectangles.

$$R = \frac{\pi}{12} \times \left(1+\tan^2\left(-\frac{\pi}{4}\right)\right) + \frac{\pi}{6} \times \left(1+\tan^2 \frac{-\pi}{6}\right) + \frac{\pi}{6}\left(1+\tan^2 0\right) + \frac{\pi}{12} \times \left(1+\tan^2 \frac{\pi}{6}\right)$$

$$= \frac{\pi}{12} \times 2 + \frac{\pi}{6}\left(1+\frac{1}{3}\right) + \frac{\pi}{6}(1) + \frac{\pi}{12} \times \left(1+\frac{1}{3}\right)$$

$$= \frac{\pi}{6} + \frac{2\pi}{9} + \frac{\pi}{6} + \frac{\pi}{9} = \frac{2\pi}{3}$$

Exercise 4. Find the area under the curve $y = x^3$ from 0 to 1 using subintervals of equal length and taking x_i^* to be the left end points. n nectangles

Length of each nectangle : $\frac{1}{n}$

$$x_i = 0 + \frac{i}{n} = \frac{i}{n}$$

$$x_i^* = \frac{i-1}{n}$$

$$A_n^L = \sum_{i=1}^{n} \frac{1}{n} \times \left(\frac{i-1}{n}\right)^3 = \frac{1}{n^4} \sum_{i=1}^{n} (i-1)^3 = \frac{1}{n^4}\left(0^3 + 1^3 + \dots + (n-1)^3\right)$$

$$\left(0^3 + 1^3 + \dots + (n-1)^3\right) = \frac{1}{n^4}\sum_{i=1}^{n-1} i^3$$

$$= \frac{1}{n^4}\left(\frac{(n-1)^2 n^2}{4}\right) = \frac{(n-1)^2}{4n^2}$$

$$\boxed{\lim_{n \to \infty} A_n^L = \frac{1}{4}}$$

Same question with x_i^* being the right endpoint.

$$A_n^R = \frac{1}{n} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^3 = \frac{1}{n^4} \frac{(n(n+1))^2}{4} = \frac{(n+1)^2}{4n^2}$$

$$\lim_{n \to \infty} A_n^R = \frac{1}{4}$$

Exercise 5. Express each definite integral as a limit of a Riemann sum:

1. $\int_1^5 \frac{1}{1+x^2} dx.$ $= \lim_{n \to \infty} \sum_{i=1}^{n} \frac{4}{n} \left(\frac{1}{1+\left(1+\frac{4i}{n}\right)^2} \right)$

2. $\int_{-1}^1 \cos x dx.$ $= \lim_{n \to \infty} \sum_{i=1}^{n} \frac{2}{n} \cos\left(-1 + \frac{2i}{n}\right)$

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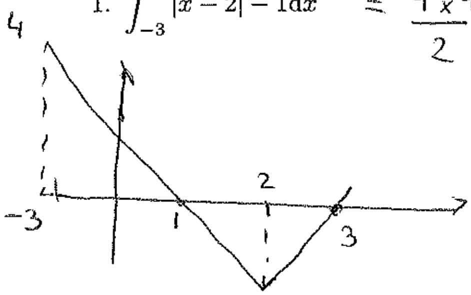
Exercise 6. Express the limit as a definite integral

1. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\left(8\left(1+\frac{3i}{n}\right)^7 - 4\right)$. $= \int_1^4 8(x)^7 - 4\, dx$

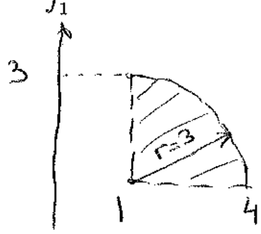
2. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{5}{n}\left(\sqrt{1+\left(3+\frac{5i}{n}\right)^2}\right)$. $= \int_3^8 \sqrt{1+x^2}\, dx$

Exercise 7. Evaluate each integral by interpreting it in terms of areas

1. $\int_{-3}^{3} |x-2| - 1 dx$ $= \frac{4 \times 4}{2} - \frac{2 \times 1}{2} = 8 - 1 = \boxed{7}$



2. $\int_{1}^{4} \sqrt{9-(x-1)^2} dx$ $= \frac{\pi \times 3^2}{4} = \frac{9}{4}\pi$



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Exercise 8. Are the functions

$$f(x) = \begin{cases} 2x+5, & 0 \leqslant x \leqslant 1 \\ 3x-1, & 1 < x \leqslant 2 \end{cases}$$

and

$$g(x) = \begin{cases} (x-1)^{-1}, & x \neq 1 \\ 2 & x = 1 \end{cases}$$

integrable on the interval $[0,2]$?

f is integrable because $2x+5$ is integrable on $[0,1]$ (two continuous functions)
and $3x-1$ ——— $[1,2]$

g is not integrable on $[0,1]$ because $\lim_{x \to 1^-} g(x) = -\infty$

Exercise 9. Evaluate the following integrals

1. $\int_0^3 3x^2 - 6x + 1dx.$ $= x^3 - 3x^2 + x \Big|_0^3 = 27 - 27 + 3 - 0 = 3$

2. $\int_0^{\sqrt{3}} \frac{1}{1+x^2}dx.$ $= \text{Arctan}\sqrt{3} - \text{Arctan}\,0 = \frac{\pi}{3}$

3. $\int_1^{\pi} \sin x dx.$ $= -\cos\pi + \cos 1 = +1 + \cos 1$

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4. $\displaystyle\int_{\ln(5)}^{\ln(2)} e^x \, dx.$ $= e^x \Big|_{\ln 5}^{\ln 2} = 2 - 5 = \boxed{-3}$

5. $\displaystyle\int_1^9 \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right) dx.$ $= \dfrac{2x^{3/2}}{3} - 2x^{1/2}\Big|_1^9 = \dfrac{2}{3}9^{3/2} - 2(9)^{1/2} - \dfrac{2}{3} + 2$

$= 2 \times 9 - 6 - \dfrac{2}{3} + 2 = \boxed{14 - \dfrac{2}{3}}$

6. $\displaystyle\int_{-5}^{3} 12x^2 \, |x-1| \, dx.$

$|x-1| = x-1$ for $x \geq 1$

$|x-1| = -(x-1)$ for $x < 1$

$\displaystyle\int_{-5}^{3} 12x^2 |x-1| dx = \int_{-5}^{1} -12x^2(x-1)dx + \int_1^3 12x^2(x-1)dx$

$= -3x^4 + 4x^3\Big|_{-5}^{1} + 3x^4 - 4x^3\Big|_1^3 = -3+4+3(5^4)+4(5^3) + 3^5 - 4(3)^3 - 3 + 4$

$= \boxed{2 + 3(5^4) + 4(5^3) + 3^5 - 4(3^3)}$

7. $\displaystyle\int_{\pi}^{\pi} \frac{\sin x}{x^2 + 2\sqrt{1+x^2}} dx.$ $\boxed{= 0}$

Exercise 10. Find the derivative of

1. $\int_1^{x^2} \cos t^2 dt$.

Let $F(t)$ be an antiderivative of $\cos t^2$. $(F'(t) = \cos t^2)$

$$\int_1^{x^2} \cos t^2 dt = F(x^2) - F(1)$$

chain rule $\Rightarrow$ $(F(x^2))' = 2x \; F'(x^2) = 2x \; \cos((x^2)^2) = 2x \cos x^4$

The derivative is $\boxed{2x \cdot \cos(x^4)}$

2. $\int_{\cos x}^{5} \frac{1}{t^4+1} dt$.

Let $F(t)$ be an antiderivative of $\frac{1}{t^4+1}$

$$F'(t) = \frac{1}{t^4+1}$$

$$\int_{\cos x}^{5} \frac{1}{t^4+1} dt = F(5) - F(\cos x)$$

$$\frac{d}{dx}\left(\int_{\cos x}^{5} \frac{1}{t^4+1} dt \right) = 0 - (-\sin x) \; F'(\cos x)$$

$$= \boxed{\sin x \; \frac{1}{\cos^4 x + 1}}$$