Texas A&M University

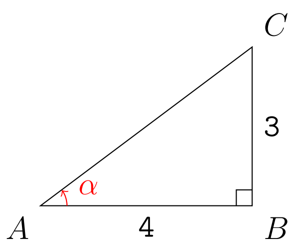
WIR 1
Appendix D, Section 1.1

1 Appendix D

1. If $\tan\alpha = \frac{3}{4}$ and $0 \leqslant \alpha \leqslant \frac{\pi}{2}$,

   find $\cos\alpha$, and $\sin\alpha$.

   Answer. Find a right triangle such that the ratio of the two sides of the right angle is $\frac{3}{4}$.



Pythagorean Theorem:

$$AC^2 = AB^2 + BC^2 = 4^2 + 3^2 = 25.$$

Therefore

$$AC = 5$$

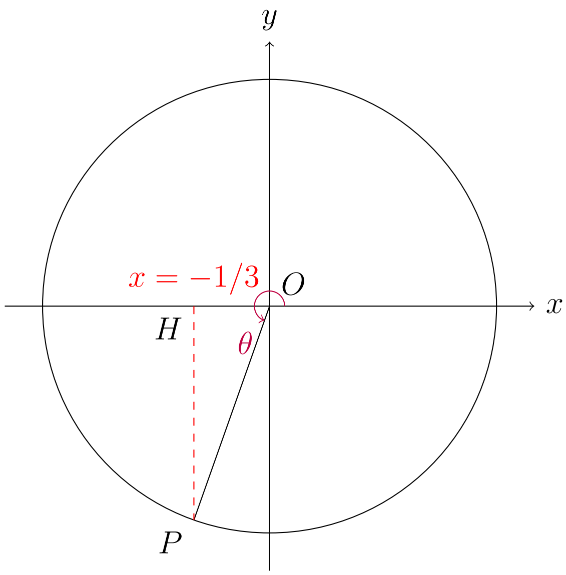
By definition $\cos\alpha = \frac{AB}{AC} = \frac{4}{5}$ and $\sin\alpha = \frac{BC}{AC} = \frac{3}{5}$.

$$\boxed{\cos\alpha = \frac{4}{5} \quad \sin\alpha = \frac{3}{5}.}$$

2. If $\cos\theta = -\dfrac{1}{3}$ and $\pi \leqslant \theta \leqslant \frac{3\pi}{2}$.

   Answer. The angle θ is represented on the trigonometric circle by the point P such that P is in the third quadrant and its x-coordinate is $\cos\theta = -\dfrac{1}{3}$.

   P is the intersection of the trigonometric circle and the vertical line $x = -\dfrac{1}{3}$ that lies in the third quadrant. Its coordinates are $(\cos\theta = -\frac{1}{3}, \sin\theta)$.



   OHP is a right triangle. $\overline{OH} = -\frac{1}{3}$, and $OP = 1$.

$$HP^2 + OH^2 = OP^2$$

   Therefore $\overline{HP} = -\dfrac{2\sqrt{2}}{3}$.

$$\sin\theta = -\frac{2\sqrt{2}}{3} \qquad \tan\theta = 2\sqrt{2}$$

$$\sec\theta = -3 \qquad \csc\theta = \frac{3\sqrt{2}}{4} \qquad \cot\theta = \frac{\sqrt{2}}{4}$$

□

3. (a) Prove that

$$\frac{\sin(a-b)}{\cos a \cos b} = \tan a - \tan b. \tag{1}$$

Answer.

$$\frac{\sin(a-b)}{\cos a\cos b} = \frac{\sin a\cos b - \cos a\sin b}{\cos a\cos b} = \frac{\sin a\cos b}{\cos a\cos b} - \frac{\cos a\sin b}{\cos a\cos b} = \frac{\sin a}{\cos a} - \frac{\sin b}{\cos b} = \tan a - \tan b$$

□

(b) For which values of a and b do the equation (1) exist?

Answer. The expression does not exist iff $\cos a\cos b = 0$

iff $\cos a = 0$ or $\cos b = 0$

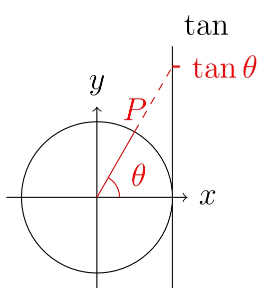
iff $a = \frac{\pi}{2} + k\pi$ for $k \in \mathbb{Z}$, or $b = \frac{\pi}{2} + k\pi$ for $k \in \mathbb{Z}$

$$\boxed{\text{The expression exists for all } (a,b) \text{ expect if } a \text{ or } b \text{ in the form } \frac{\pi}{2} + k\pi (k \in \mathbb{Z})}$$

4. Solve the following equations on the interval $[0, 2\pi]$.

(a) $0 \leqslant \tan x \leqslant 1$.

Answer.



$$\boxed{\left[0, \frac{\pi}{4}\right] \cup \left[\pi, \frac{5\pi}{4}\right]}$$

(b) $2\sin^2 - 1 = 0$.

Answer. Factor out the expression using the identity

$$\boxed{a^2 - b^2 = (a-b)(a+b)}$$

$$(\sqrt{2}\sin x - 1)(\sqrt{2}\sin x + 1) = 0.$$

A product of two factors is 0 iff at least one of the factors is zero.

iff

$$\begin{aligned} &\sqrt{2}\sin x - 1 = 0 && \textit{or} && \sqrt{2}\sin x + 1 = 0 \\ &\sin x = \frac{\sqrt{2}}{2} && && \sin x = -\frac{\sqrt{2}}{2} \\ &x = \frac{\pi}{4}, \quad \texttt{or} \quad x = \frac{3\pi}{4} && \textit{or} && x = \frac{5\pi}{4}, \quad \texttt{or} \quad x = \frac{7\pi}{4} \end{aligned}$$

$$\boxed{4 \text{ solutions: } \frac{\pi}{4}, \quad \frac{3\pi}{4} \quad \frac{5\pi}{4} \quad \frac{7\pi}{4}}$$

(c) $\sin x \cos(2x) + \sin(2x)\cos x = \frac{1}{2}$.

Answer. Use the identity

$$\boxed{\sin(a+b) = \sin a \cos b + \cos a \sin b}$$

$$\sin(x + 2x) = \frac{1}{2}$$

$$3x = \frac{\pi}{6} + 2k\pi \quad (k \in \mathbb{Z}) \text{ or } 3x = \frac{5\pi}{6} + 2k\pi \quad (k \in \mathbb{Z}).$$

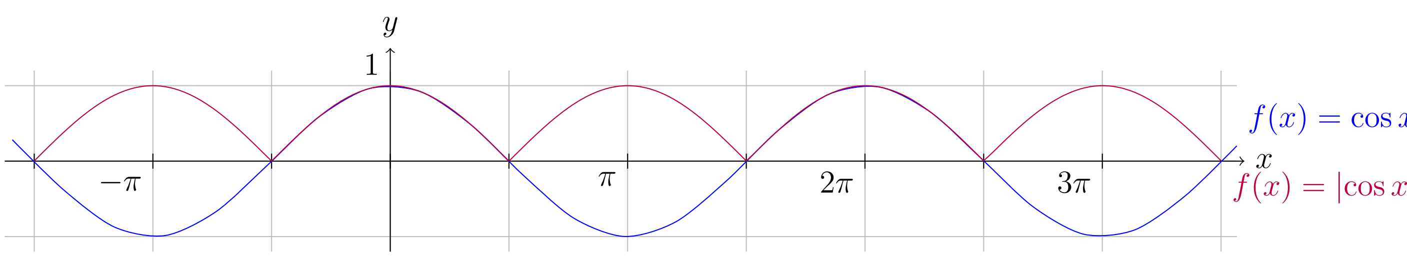
$$x = \frac{\pi}{18} + \frac{2k\pi}{3} \quad (k \in \mathbb{Z}) \text{ or } x = \frac{5\pi}{18} + \frac{2k\pi}{3} \quad (k \in \mathbb{Z}).$$

In $[0, 2\pi]$,

$$\boxed{x \in \left\{ \frac{\pi}{18}, \frac{13\pi}{18}, \frac{25\pi}{18}, \frac{5\pi}{18}, \frac{17\pi}{18}, \frac{29\pi}{18} \right\}}$$

5. Sketch the graph the function

$$y = |\cos x|.$$

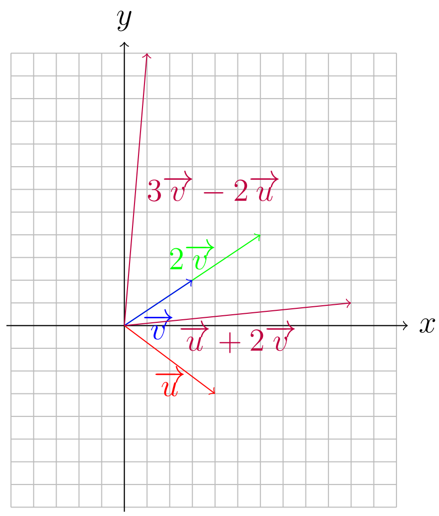


2 Section 1.1

1. Let $\overrightarrow{u} = 4\mathbf{i} - 3\mathbf{j}$ and $\overrightarrow{v} = \langle 3, 2 \rangle$.

   (a) Sketch the position vectors of $\overrightarrow{u}$ and $\overrightarrow{v}$.

(b) Represent geometrically the vectors $\vec{w} = \vec{u} + 2\vec{v}$ and the vector $\vec{s} = 3\vec{v} - 2\vec{u}$.



(c) Calculate the magnitude of $\vec{u} - \vec{v}$.

Answer. $\vec{u} - \vec{v} = \langle 4, -3\rangle - \langle 3, 2\rangle = \langle 1, -5\langle$.
$||\vec{u} - \vec{v}|| = \sqrt{1^2 + 5^2}$

$$\boxed{||\vec{u} - \vec{v}|| = \sqrt{26}}$$

(d) Find a unit vector that has the same direction as $\vec{u}$.

Answer. `A unit vector` $\vec{U}$ `with the same direction as` $\vec{u}$ `is given by`

$$\vec{U} = \frac{1}{||\vec{u}||}\vec{u}$$

$||\vec{u}|| = \sqrt{4^2 + (-3)^2} = 5$

$$\boxed{\vec{U} = \langle \frac{4}{5}, \frac{-3}{5}\rangle}$$

(e) Given $\vec{t} = \langle 5, -8\rangle$, find 2 real numbers a and b such that $\vec{t} = a\vec{u} + b\vec{v}$.

Answer. `If` $\vec{t} = a\vec{u} + b\vec{v}$ `then` $\langle 5, -8\rangle = a\langle 4, -3\rangle + b\langle 3, 2\rangle$

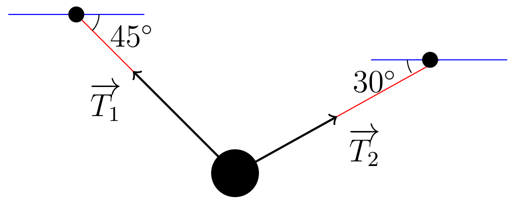
$$\langle 5, -8\rangle = \langle 4a + 3b, -3a + 2b\rangle$$

$$\begin{cases} 5 = 4a + 3b \\ -8 = -3a + 2b \end{cases}$$

`Solve the System...`

$$\boxed{a = 2, b = -1}$$

2. A 50lb weight is hanging from two 10ft long ropes as shown below:



Find the components of the tension forces and their magnitudes.

Answer. The weight is in equilibrium therefore $\vec{T_1} + \vec{T_2} + \vec{W} = \vec{0}$.
Let T_1 be the magnitude of $\overrightarrow{T_1}$ and T_2 be the magnitude of $\overrightarrow{T_2}$.
Then $\overrightarrow{T_1} = \langle -T_1 \cos 45°, T_1 \sin 45° \rangle$.
$\overrightarrow{T_2} = \langle T_2 \cos 30°, T_1 \sin 30° \rangle$.
$\overrightarrow{W} = \langle 0, -50 \rangle$.

$$\vec{0} = \overrightarrow{W} + \overrightarrow{T_1} + \overrightarrow{T_2} = \langle 0 - T_1 \frac{\sqrt{2}}{2} + T_2 \frac{\sqrt{3}}{2}, -50 + T_1 \frac{\sqrt{2}}{2} + \frac{1}{2} T_2 \rangle.$$

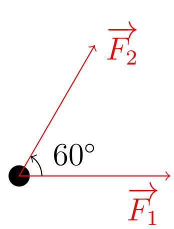
$$\begin{cases} 0 = -\frac{\sqrt{2}}{2} T_1 + \frac{\sqrt{3}}{2} T_2 \\ 0 = -50 + \frac{\sqrt{2}}{2} T_1 + \frac{1}{2} T_2 \end{cases}$$

Solve for T_1 and T_2.

$$\begin{cases} T_1 = \frac{\sqrt{3}}{\sqrt{2}} T_2 \\ 0 = -50 + \frac{\sqrt{3}}{2} T_2 + \frac{1}{2} T_2 \end{cases}$$

$$\boxed{\begin{gathered} T_1 = \frac{25(3\sqrt{2} - \sqrt{6})}{2}, \\ T_2 = 25(\sqrt{3} - 1) \\ \overrightarrow{T_1} = \langle -25 \frac{(3 - \sqrt{3})}{2}, 25 \frac{(3 - \sqrt{3})}{2} \rangle, \\ \overrightarrow{T_2} = \langle 25 \frac{(3 - \sqrt{3})}{2}, 25 \frac{(\sqrt{3} - 1)}{2} \rangle \end{gathered}}$$

3. 2 forces of 10lb each are pulling a weight as shown in the picture



Find the components of the resultant force $\overrightarrow{F}$ acting on the weight, its magnitude and the angle between $\overrightarrow{F_1}$ and $\overrightarrow{F}$.

Answer. Find the components of $\overrightarrow{F_1}$ and $\overrightarrow{F_2}$:

$\overrightarrow{F_1} = \langle 10, 0\rangle$ and $\overrightarrow{F_2} = \langle 10\cos(60°), 10\sin(60°)\rangle = \langle 5, 5\sqrt{3}\rangle$.

By definition $\overrightarrow{F} = \overrightarrow{F_1} + \overrightarrow{F_2} = \langle 15, 5\sqrt{3}\rangle$.

The magnitude of $\overrightarrow{F} = \sqrt{225+75} = 10\sqrt{3}$.

The coordinates of $\overrightarrow{F}$ are both positive, therefore the angle θ between $\overrightarrow{F_1}$ and $\overrightarrow{F}$ is in the first quadrant.

Considering the first component of $\overrightarrow{F}$, $15 = F\cos(\theta) = 10\sqrt{3}\cos(\theta)$.

$cos(\theta) = \dfrac{15}{10\sqrt{3}} = \dfrac{\sqrt{3}}{2}$ and $\theta = \dfrac{\pi}{6}$.

$$\boxed{\overrightarrow{F} = \langle 15, 5\sqrt{3}\rangle, \quad \left\|\overrightarrow{F}\right\| = 10\sqrt{3}, \quad \theta = \frac{\pi}{6}}$$

4. A pirate walks in the west direction on a ship at a speed of 2 mi/h. His ship is moving north at a speed of 10mi/h. Find the speed and the direction of the pirate relative to the surface of the water.

   Answer. Let $\overrightarrow{V}_{P/S}$ be the velocity of the pirate (P) relative to the ship (S).

   Let $\overrightarrow{V}_{S/G}$ be the velocity of the ship relative to the ground (G).

   Then the velocity of the pirate relative to the ground

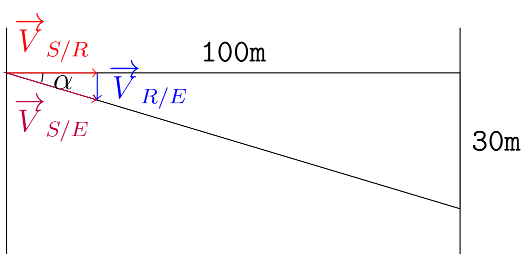
   $$\overrightarrow{V}_{P/G} = \overrightarrow{V}_{P/S} + \overrightarrow{V}_{S/G}.$$

   $\overrightarrow{V}_{P/S} = \langle -2, 0\rangle$, $\overrightarrow{V}_{S/G} = \langle 0, 10\rangle$ therefore $\overrightarrow{V}_{P/G} = \langle -2, 10\rangle$.

   Conclusion: The speed of the pirate relative to the ground is $\sqrt{2^2+10^2} = 2\sqrt{26}$mph. The direction is the direction of the vector $\langle -2, 10\rangle$.

   □

5. A swimmer heads directly across the river and swims at a constant speed of 20 m/min. He reaches the other side of the river 30m downstream. What is the speed of the current?



   Answer.

Let $\overrightarrow{V}_{S/R}$ be the velocity of the swimmer S relative to the river R and $\overrightarrow{V}_{R/E}$ is the velocity of the river related to earth.

The velocities are constant therefore the swimmer moves in straight line.

$\tan\alpha = \frac{30}{100} = \frac{||\overrightarrow{V}_{R/E}||}{||\overrightarrow{V}_{S/R}||}$.

$$\boxed{\left|\left|\overrightarrow{V}_{S/R}\right|\right| = 6\text{m/min.}}$$