WIR 2
Sections 1.2, 1.3, 2.2

1 Section 1.2

1. Find $\vec{a} \cdot \vec{b}$

   (a) if $||\vec{a}|| = 3$, $\left|\left|\vec{b}\right|\right| = 2$, and the angle between $\vec{a}$ and $\vec{b}$ is $\frac{4\pi}{3}$.

   Answer. By definition, $\vec{a} \cdot \vec{b} = ||\vec{a}|| \left|\left|\vec{b}\right|\right| \cos\theta$.

   $\cos\frac{4\pi}{3} = \frac{-1}{2}$.

   $$\boxed{\vec{a} \cdot \vec{b} = -3}$$

   (b) if $\vec{a} = \langle 2, -3\rangle$ and $\vec{b} = 4\mathbf{i} + 5\mathbf{j}$.

   Answer.

   $$\begin{aligned}\vec{a} \cdot \vec{b} &= x_a x_b + y_a y_b \\ &= 8 - 15\end{aligned}$$

   $$\boxed{\vec{a} \cdot \vec{b} = -7}$$

2. Find the value(s) of x such that the vectors $\vec{u} = \langle 2x, -x\rangle$ and $\vec{v} = \langle 3, x\rangle$ are orthogonal.

   Answer.

   Remark: $\vec{0}$ is orthogonal to any vector.

   $\vec{u} \perp \vec{v}$ iff $\vec{u} \cdot \vec{v} = 0$

   iff $3(2x) + (-x)(x) = 0$

   iff $6x - x^2 = 0$

   $$\boxed{x = 0 \text{ or } 6}$$

3. Find the value(s) of x such that $\vec{s} = \langle 1, x\rangle$ and $\vec{t} = \langle 3, 4\rangle$ are parallel.

   Answer. $\vec{s}$ and $\vec{t}$ are parallel iff $\vec{s} \cdot \vec{t} = \pm ||\vec{s}|| \cdot \left|\left|\vec{t}\right|\right|$.

   Evaluate the dot product: $\vec{s} \cdot \vec{t} = 3 + 4x$.

   Evaluate the product of norms: $||\vec{s}|| \cdot \left|\left|\vec{t}\right|\right| = 5\sqrt{1 + x^2}$.

   $\vec{s} \cdot \vec{t} = \pm ||\vec{s}|| \cdot \left|\left|\vec{t}\right|\right|$ iff $3 + 4x = \pm 5\sqrt{1 + x^2}$.

   iff $(3 + 4x)^2 = 25(1 + x^2)$. (Square the equation)

   iff $16x^2 + 24x + 9 = 25 + 25x^2$.

   iff $9x^2 - 24x + 16 = 0$.

   iff $(3x - 4)^2 = 0$

   $$\boxed{x = \frac{4}{3}}$$

4. Find the work done by a force of 30lb acting in the direction N30°W in moving an object 20ft West.

   Answer. $W = \overrightarrow{F} \cdot \overrightarrow{D}$ where $\overrightarrow{F}$ is the force and $\overrightarrow{D}$ is the displacement vector.

   $W = \left|\left|\overrightarrow{F}\right|\right| \cdot \left|\left|\overrightarrow{D}\right|\right| \cos\theta$. Here $\theta = 60°$.

   $W = 30 * 20 * \frac{1}{2}$

   $$\boxed{W = 300\text{ft-lb}}$$

5. A 10lb block slide down a straight ramp for the initial point $(0, 15)$ to the final position $(7, 0)$. Find the work done by the gravity force on the block.

   Answer. Again $W = \overrightarrow{F} \cdot \overrightarrow{D}$.

   Find the components: $\overrightarrow{F} = \langle 0, -10\rangle$, $\overrightarrow{D} = \langle 7-0, 0-15\rangle = \langle 7, -15\rangle$

   $W = \overrightarrow{F} \cdot \overrightarrow{D} = 0 * 7 - 10 * (-15)$

   $$\boxed{W = 150\text{ft=lb}}$$

6. Find the distance from the point $P(1, 3)$ to the line $y = 2x - 1$.

   Answer.

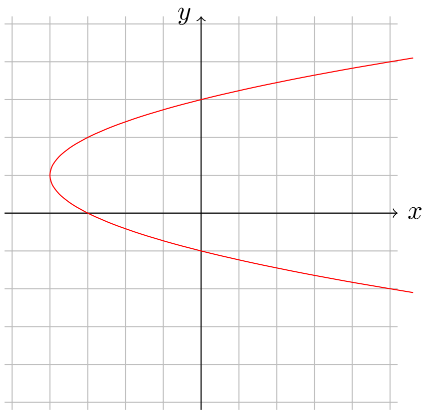
   - Find 2 points on the line:
     $A(0, -1)$ and $B(1, 1)$.
   - Evaluate the vector $\overrightarrow{AB}$ and its orthogonal complement $\overrightarrow{u}$:
     $\overrightarrow{AB} = \langle 1, 2\rangle$ $\overrightarrow{u} = \overrightarrow{AB}^{\perp} = \langle -2, 1\rangle$.
   - Evaluate the components of the vector $\overrightarrow{AP}$ (You may use any vector from one point on the line to P, I chose A)
     $\overrightarrow{AP} = \langle 1-0, 3-(-1)\rangle = \langle 1, 4\rangle$.
   - The distance is the scalar projection of $\overrightarrow{AP}$ on $\overrightarrow{u}$.
     $$d = Comp_{\overrightarrow{u}}(\overrightarrow{AP}) = \frac{\overrightarrow{u} \cdot \overrightarrow{AP}}{||\overrightarrow{u}||} = \frac{1 * (-2) + 4 * 1}{\sqrt{1^2 + (-2)^2}} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$$

   $$\boxed{d = \frac{2\sqrt{5}}{5}}$$

2 Section 1.3

1. For each parametric equation,

   - sketch the curve.
   - Eliminate the parameter to find a Cartesian equation.
   - $x(t) = t^2 - 4 \quad y(t) = 1 - t$.
     Answer. To sketch the curve, pick values of t and calculate $x(t)$ and $y(t)$.



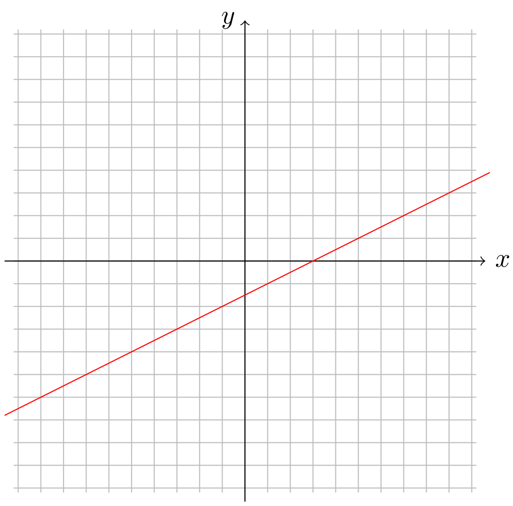
$1 - t = y$ therefore $t = 1 - y$

$x = t^2 - 1 = (1 - y)^2 - 4 = 1 - 2y + y^2 - 4 = y^2 - 2y - 3$

$$\boxed{x = y^2 - 2y - 3}$$

- $x(t) = 2t - 7, \quad y(t) = t - 5.$

  Answer.



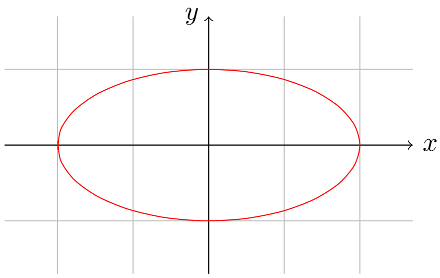
$y = t - 5$ therefore $t = y + 5$. $\quad x = 2t - 7 = 2(y + 5) - 7 = 2y + 3$

There are infinitely many Cartesian equations for that line $y = \frac{x-3}{2}$ is correct too.

$$\boxed{x = 2y + 3}$$

- $x(t) = 2\cos t \quad y(t) = \sin t.$

  Answer.



The idea is to use the identity $\cos^2 t + \sin^2 t = 1$

$\left(\frac{x}{2}\right)^2 + y^2 = \cos^2 t + \sin^2 t = 1$ It is a Cartesian equation for the ellipse. Another one:

$$\boxed{x^2 + 4y^2 = 4}$$

2. Find a vector equation, a parametric equation, and a Cartesian equation for

   (a) the line passing through the points $A(1, -2)$ and $B(3, 4)$.

   Answer. $\overrightarrow{AB} = \langle 2, 6\rangle$. The line L_1 passes through A and is parallel to $\overrightarrow{AB}$.

   A vector equation is $\boxed{\overrightarrow{r}(t) = \overrightarrow{0A} + t\overrightarrow{AB} = \langle 1 + 2t, -2 + 6t\rangle}$

   A parametric equation is $\boxed{x(t) = 1 + 2t, \quad y(t) = -2 + 6t.}$

   $x = 1 + 2t$ therefore $2t = x - 1$.

   A Cartesian equation:

   $y = -2 + 6t = -2 + 3(x - 1) = -2 + 3x - 3 = 3x - 5$

   $\boxed{y = 3x - 5.}$

   □

   (b) the line passing through the point $C(1, 3)$, and parallel to the vector $\overrightarrow{v} = \langle 2, 5\rangle$.

   Answer. A vector equation: $\boxed{r(t) = \overrightarrow{OC} + t\overrightarrow{v} = \langle 1 + 2t, 3 + 5t\rangle.}$

   A parametric equation: $\boxed{x(t) = 1 + 2t, \quad y(t) = 3 + 5t}$.

   $x = 1 + 2t$ therefore $t = \dfrac{x - 1}{2}$.

   $y = 3 + 5t = 3 + 5\dfrac{x - 1}{2} = \dfrac{6 + 5x - 5}{2} = \dfrac{1 + 5x}{2}$

   Possible Cartesian equations:

$$\boxed{y = \frac{1 + 5x}{2}, \quad \text{or } 2y = 1 + 5x, \quad \text{or } x = \frac{2y - 1}{5}, \quad \text{or } 2y - 5x = 1, \ldots}$$

   □

   (c) the line passing through the point $D(-1, 2)$, and orthogonal to $\overrightarrow{w} = \langle 3, 5\rangle$.

   Cartesian equation: Let $M(x, y)$ be a point on the line $\overrightarrow{DM} \perp \overrightarrow{w}$, therefore $\overrightarrow{DM} \cdot \overrightarrow{w} = 0$

   $\overrightarrow{DM} \cdot \overrightarrow{w} = 3(x + 1) + 5(y - 2) = 0$

   $\boxed{3x + 5y - 7 = 0.}$

   Vector equation: The line is orthogonal to $\overrightarrow{w}$ therefore the line is parallel to the orthogonal complement of $\overrightarrow{w}^{\perp} = \langle -5, 3\rangle$.

$\boxed{\overrightarrow{r}(t) = \overrightarrow{OD} + t\overrightarrow{w}^{\perp} = \langle -1-5t, 2+3t\rangle}$

Parametric equation from vector equation $\boxed{x(t) = -1-5t, \quad y(t) = 2+3t.}$

Parametric equation from the Cartesian equation: Let $x = t$, then $y = \dfrac{7-3x}{5} = \dfrac{7-3t}{5}$.

$\boxed{x(t) = t, \quad y(t) = \dfrac{7-3t}{5}}$

3. Are the lines

$$L_1 : \overrightarrow{r}_1(t) = \langle 4+t, 1+2t\rangle \quad L_2 : \overrightarrow{r}_2(t) = \langle 5-4t, -2+2t\rangle$$

orthogonal, parallel or neither.

In case L_1 and L_2 are not parallel, find the coordinates of the intersection point of L_1 and L_2.

Answer. $\overrightarrow{r}_1(t) = \langle 4,1\rangle + t\langle 1,2\rangle$. Therefore L_1 is the line passing through the point $P(4,1)$ and parallel to the vector $\overrightarrow{v} = \langle 1,2\rangle$.

Similarly, $\overrightarrow{r}_2(t) = \langle 5,-2\rangle + t\langle -4,2\rangle$. L_2 is the line passing through the point Q and parallel to the vector $\overrightarrow{v} = \langle -4,2\rangle$

To determine whether the lines are orthogonal, i.e. $\overrightarrow{u} \perp \overrightarrow{v}$, check if their dot product is 0.

$\overrightarrow{u} \cdot \overrightarrow{v} = -4+4 = 0$. $\boxed{\text{The lines are orthogonal.}}$

Therefore they intersects.

2 Methods to find the intersection I

- Method 1: I belongs to L_1 therefore there exists t such that $\overrightarrow{r}_1(t) = \overrightarrow{OI}$.
  I is a point of L_2 too, therefore there exists s such that $\overrightarrow{r}_2(s) = \overrightarrow{OI}$.
  Our goal now is to find t and s.
  $\langle 4+t, 1+2t\rangle = \langle 5-4s, -2+2s\rangle$.
  Therefore
  $\begin{cases} 4+t = 5-4s \\ 1+2t = -2+2s \end{cases}$ $\begin{cases} t = 5-4s-4 = 1-4s \\ 1+2(1-4s) = -2+2s \end{cases}$
  $\begin{cases} t = 1-4s \\ 5 = 10s \end{cases}$ $\begin{cases} s = 0.5 \\ t = 1-4s = -1 \end{cases}$
  The coordinates of I are given by $\overrightarrow{r}_1(-1) = \overrightarrow{r}_2(.5)$
  $\boxed{I(3,-1).}$
- Method 2: Find Cartesian equation for L_1
  $t = x-4$, $y = 1+2(x-4) = 2x-7$
  $L_1 : y = 2x-7$.
  Find a Cartesian equation for L_2:
  $t = \dfrac{5-x}{4}$, $y = -2+2\dfrac{5-x}{4} = \dfrac{1-x}{2}$
  $L_2 : y = \dfrac{1-x}{2}$.
  The coordinates (x,y) of I are solution of
  $\begin{cases} y = 2x-7 \\ y = \frac{1-x}{2} \end{cases}$
  $\begin{cases} y = 2x-7 \\ 4x-14 = 1-x \end{cases}$ $\begin{cases} x = 3 \\ y = 2x-7 = -1 \end{cases}$
  The coordinates of I are $\boxed{I(3,-1).}$

□

4. A marble is moving in the xy-plane. Its position at time t is given by

$$x(t) = 2t + 5 \quad y(t) = 8t - t^2.$$

(a) Find the position of the marble at time $t = 3$.
Answer. At $t = 3$, $x(3) = 2 * 3 + 5 = 11, \quad y(3) = 8 * 3 - 9 = 15$

$$\boxed{(11, 15)}$$

(b) The line $y = 15$ is traced on the plane. At what time(s) does the marble cross the line?
Answer. The marble crosses the line $y = 15$ when $y(t) = 15$.
$y(t) = 15$ iff $8t - t^2 = 15$ iff $t = 3$ or $t = 5$

$$\boxed{t = 3, \text{ and } t = 5}$$

(c) Does the marble pass through the point $(9, 12)$?
Answer. It passes through the point $(9, 12)$ if there exists t such that $x(t) = 9$ and $y(t) = 12$.
$\begin{cases} 2t + 5 = 9 \\ 8t - t^2 = 12 \end{cases}$ $\begin{cases} t = 2 \\ 8 * 2 - 2^2 = 12 \end{cases}$

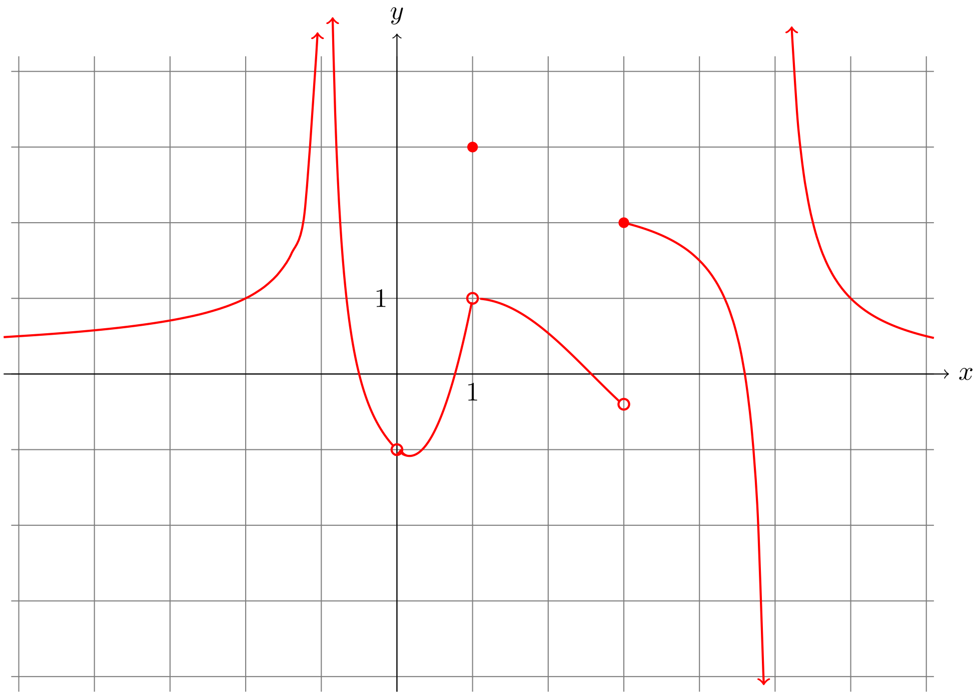
$$\boxed{\text{Yes, at } t = 2}$$

(d) Does the marble pass through the point $(7, 12)$?
Answer. It passes through the point $(7, 12)$ if there exists t such that $x(t) = 7$ and $y(t) = 12$. $\begin{cases} 2t + 5 = 7 \\ 8t - t^2 = 12 \end{cases}$ $\begin{cases} t = 1 \\ 8 * 1 - 1^2 = 7 \neq 12 \end{cases}$

$$\boxed{No}$$

3 Section 2.2

1. Given the function f defined by its graph,



(a) What is the domain of f?

Answer. The function is not defined at $x = -1$, $x = 0$, and $x = 5$.
The domain in interval notation:

$$\boxed{(-\infty, -1) \cup (-1, 0) \cup (0, 5) \cup (5, \infty)}$$

(b) Complete the following equalities:

$\lim_{x\to -1^-} f(x) = \infty$	$\lim_{x\to -1^+} f(x) = \infty$	$\lim_{x\to -1} f(x) = \infty$	$f(-1) = DNE$
$\lim_{x\to 0^-} f(x) = -1$	$\lim_{x\to 0^+} f(x) = -1$	$\lim_{x\to 0} f(x) = -1$	$f(0) = DNE$
$\lim_{x\to 1^-} f(x) = 1$	$\lim_{x\to 1^+} f(x) = 1$	$\lim_{x\to 1} f(x) = 1$	$f(1) = 3$
$\lim_{x\to 3^-} f(x) = 0.4$	$\lim_{x\to 3^+} f(x) = 2$	$\lim_{x\to 3} f(x) = DNE$	$f(3) = 2$
$\lim_{x\to 5^-} f(x) = -\infty$	$\lim_{x\to 5^+} f(x) = +\infty$	$\lim_{x\to 5} f(x) = DNE$	$f(5) = DNE$

2. Find the asymptotes and holes of the function $f(x) = \dfrac{x-1}{x^2-3x+2}$.

Answer. The function f is defined all the real numbers except for $x = 1$ and $x = 2$.

for $x \neq 1$ and $x \neq 2$, $f(x) = \dfrac{x-1}{(x-1)(x-2)} = \dfrac{1}{x-2}$

There is a hole at $x = 1$ and an asymptote at $x = 2$.

$\lim_{x\to 2^-} f(x) = -\infty$ $\lim_{x\to 2^+} f(x) = +\infty$

□