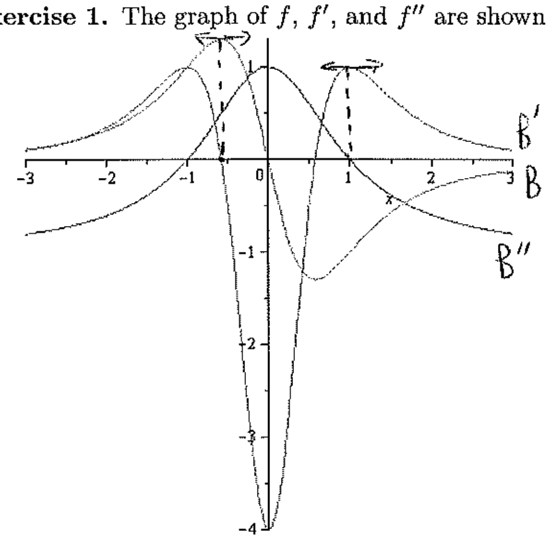
Sections 3.8-3.10

Exercise 1. The graph of f, f', and f'' are shown below. Label which is which.



Use the fact that $f'(x) = 0$ when the graph of f has a horizontal tangent

The derivative of the green function is the yellow function
The derivative of the yellow function is the red function.

Therefore $\begin{cases} \text{graph}(f) \text{ is green} \\ \text{graph}(f') \text{ is yellow} \\ \text{graph}(f'') \text{ is red.} \end{cases}$

Exercise 2. Calculate the second derivative of

$$f(x) = \frac{x+1}{x-1}, \quad g(x) = \sqrt{2x^3 - 3x^2}.$$

$$f'(x) = \frac{(x-1)-(x+1)}{(x-1)^2} = \frac{-2}{(x-1)^2}$$

$$f''(x) = (-2)(-2)(x-1)^{-3}$$

$$\boxed{f''(x) = 4(x-1)^{-3} = \frac{4}{(x-1)^3}}$$

$$g'(x) = \frac{6x^2 - 6x}{2\sqrt{2x^3-3x^2}} = \frac{3(x^2-x)}{\sqrt{2x^3-3x^2}}$$

$$g''(x) = \frac{3\left((2x-1)\sqrt{2x^3-3x^2} - \frac{(x^2-x)(6x^2-6x)}{\sqrt{2x^3-3x^2}}\right)}{(2x^3-3x^2)}$$

Exercise 3. Find $\frac{d^2y}{dx^2}$ for

$$y^3 + x^3 = 8$$

$$3y^2\frac{dy}{dx} + 3x^2 = 0 \qquad \frac{dy}{dx} = \frac{-x^2}{y^2}$$

$$y^2\frac{dy}{dx} + x^2 = 0$$

$$2y\left(\frac{dy}{dx}\right)^2 + y^2\frac{d^2y}{dx^2} + 2x = 0$$

$$\frac{dy}{dx} = \frac{-2x - 2y\left(\frac{dy}{dx}\right)^2}{y^2} = \frac{-2x - 2y\left(\frac{x^4}{y^4}\right)}{y^2} = -2\left(\frac{xy^4 + yx^4}{y^6}\right)$$

Exercise 4. Find a formula for $\frac{d^n f}{dx^n}$ for $f(x) = \frac{3}{5+x}$.

$f'(x) = 3\frac{(-1)}{(5+x)^2} = 3(-1)(5+x)^{-2}$

$f''(x) = 3(-1)(-2)(5+x)^{-3}$

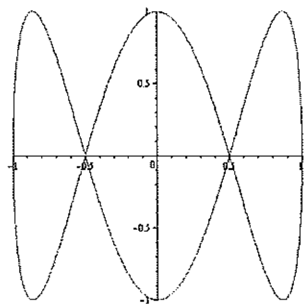
$f'''(x) = 3(-1)(-2)(-3)(5+x)^{-4}$

$f^{(n)}(x) = 3(-1)(-2)(-3)\dots(-n)(x+5)^{-n-1}$

$\boxed{f^{(n)}(x) = 3(-1)^n\, n!\,(x+5)^{-n-1}}$

Exercise 5. The trajectory of a particle is given by

$$\vec{r}(t) = \langle \sin t, \cos 3t\rangle, \quad t \in [0, 2\pi]$$



1. Find the acceleration at $t = \frac{\pi}{4}$.

$\vec{r}\,'(t) = \langle \cos t, -3\sin 3t\rangle$

$\vec{r}\,''(t) = \langle -\sin t, -9\cos 3t\rangle$

acceleration at t.

$\vec{r}\,''(\frac{\pi}{4}) = \langle -\sin\frac{\pi}{4}, -9\cos(\frac{3\pi}{4})\rangle$

$\boxed{\vec{r}\,''(\frac{\pi}{4}) = \langle -\frac{\sqrt{2}}{2}, \frac{9\sqrt{2}}{2}\rangle}$

2. Find the equations of the tangent lines at the point $(0.5, 0)$.

* Find the parameter t.

$\begin{cases} \sin t = 0.5 \\ \cos 3t = 0 \end{cases}$ $\begin{cases} t = \frac{\pi}{6} \text{ or } t = \frac{5\pi}{6} \\ 3t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, \frac{11\pi}{2} \end{cases}$

$\begin{cases} t = \frac{\pi}{6} \text{ or } t = \frac{5\pi}{6} \\ \text{and} \\ t = \frac{\pi}{6} \text{ or } \frac{\pi}{2} \text{ or } \frac{5\pi}{6} \text{ or } \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6} \end{cases}$

2 times: $t = \frac{\pi}{6}$ and $t = \frac{5\pi}{6}$

slope $= \frac{-3\sin 3t}{\cos t}$

$\text{slope}_{\frac{\pi}{6}} = \frac{-3\times 2}{\sqrt{3}} = -2\sqrt{3}$

$\text{slope}_{\frac{5\pi}{6}} = \frac{+3\times 2}{\sqrt{3}} = +2\sqrt{3}$

slope-point equation at $t = \frac{\pi}{6}$
$y = -2\sqrt{3}(x - 0.5)$
slope point equation at $t = \frac{5\pi}{6}$
$y = 2\sqrt{3}(x - 0.5)$

3. At what point does the curve have an horizontal tangent, or a vertical tangent?

horizontal tangent if $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$

vertical tangent if $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$.

$\frac{dx}{dt} = 0 = \cos t$

$t = \frac{\pi}{2}$, $t = \frac{3\pi}{2}$

vertical tangent $(1, 0), (-1, 0)$

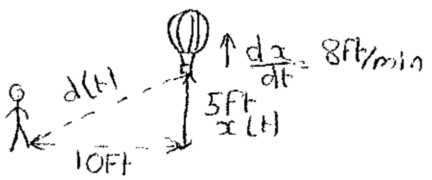
$\frac{dy}{dt} = 0 = -3\sin(3t)$ $\quad 3t = 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi, 6\pi$

$t = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, 2\pi$

$t \in [0, 2\pi]$, $3t \in [0, 6\pi]$ $\quad t = 0$

horizontal tangent $(0, 1)$ $(\frac{\sqrt{3}}{2}, -1), (\frac{\sqrt{3}}{2}, 1), (0, -1), (-\frac{\sqrt{3}}{2}, 1)(-\frac{\sqrt{3}}{2}, -1)$

Exercise 9. (fall 2011) An observer stands 10 feet from the base of a balloon launching point. At the instant the balloon has risen vertically 5 feet, the height of the balloon is increasing at a rate of 8 feet per minute. How fast is the distance from the observer to the balloon changing at this same instant? Assume the balloon starts on the ground and rises vertically.



$x(t)$ height of the ballon at time t

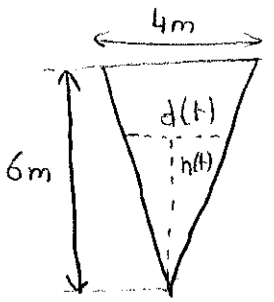
$d(t)$: distance from the observer to the ballon.

$$d^2(t) = x^2(t) + 10^2 = 100 + x^2(t)$$

$$2\, d(t) . d'(t) = 2\, x(t)\, x'(t)$$

$$d'(t) = \frac{x(t)}{d(t)} x'(t) = \frac{5 \times 8}{\sqrt{125}} = \frac{8}{\sqrt{5}} \approx 2.24 \text{ Ft/min}$$

Exercise 10. (17p219) Water is leaking out of an inverted conical tank at a rate of 10,000cm^3/min at the same time that water is being pumped into the tank at constant rate. The tank has height 6m and the diameter at the top is 4m. If the water level is rising at a rate of 20cm/min when the height of water is 2m, find the rate at which water is being pumped into the tank.



$h(t)$ height of water at time t

$V(t)$ volume of water at time t.

$d(t)$ diameter of the surface of water at time t

$$\frac{dV}{dt} = \text{rate in} - \text{rate out} = \text{rate}_{in} - \cancel{1}0.1 \text{ m}^3/\text{min}.$$

$$V(t) = \frac{1}{3} \pi \frac{h\, d^2}{4} = \frac{\pi}{12} h\, d^2$$

$$\frac{d}{h} = \frac{4}{6} = \frac{2}{3} \quad \text{therefore} \quad \boxed{d = \frac{2}{3} h}$$

$$V(t) = \frac{\pi}{12} h \left(\frac{2}{3} h\right)^2 = \frac{\pi}{12} \times \frac{4}{9} h^3 = \frac{\pi}{27} h^3$$

$$\frac{dV}{dt} = \text{rate}_{in} - 0.1 = \frac{\pi}{27} \times 3h^2 \frac{dh}{dt} = \frac{\pi}{9} h^2 \frac{dh}{dt}$$

$$\frac{dh}{dt} = 0.2 \text{ m/min}$$

$$h = 2\text{m}$$

$$\frac{dV}{dt} \approx 0.28 \text{ m}^3/\text{min}$$

$$\boxed{\text{rate}_{in} = 0.38 \text{m}^3/\text{min}}$$

Exercise 6. Given the parametric curve,

$$x(t) = t(t^2 - 9), \quad y(t) = t^2 - 1,$$

1. At what point does the curve cross itself. Find the equations of the tangent lines at that point.

$\begin{cases} x(s) = x(t) \\ y(s) = y(t) \\ t \neq s \end{cases}$ $\begin{cases} s(s^2-9) = t(t^2-9) \\ s^2-1 = t^2-1 \\ s \neq t \end{cases}$ $\begin{cases} s^2 = t^2 \\ s(t^2-9) - t(t^2-9) = 0 \\ t \neq s \end{cases}$

$\begin{cases} t = -s \\ (t^2-9)(s-t) = 0 \\ t \neq s \end{cases}$ $\begin{cases} t = -s \\ t^2 - 9 = 0 \end{cases}$ $\begin{cases} t = \pm 3 \\ s = \mp 3 \end{cases}$

$\boxed{\text{point given by } x(3) = 0 \ \ y(3) = 8 \quad (0,8) \text{ at } t=3 \text{ and } t=-3}$

slope $= \dfrac{y'}{x'} = \dfrac{2t}{3t^2-9}$

slope$_{t=3} = \dfrac{1}{3}$ slope$_{t=-3} = \dfrac{-1}{3}$

slope point equation $\boxed{y = \frac{1}{3}x + 8}$ slope point equation $\boxed{y = -\frac{1}{3}x + 8}$

2. At what point(s) is the curve parallel to the line

$$x(t) = 1 + 3t, \quad y(t) = 4 + t$$

slope of the line is given by $\dfrac{y'(t)}{x'(t)} = \dfrac{1}{3}$

slope of the tangent line is $\dfrac{2t}{3t^2-9}$

the tangent is parallele to the line if $\dfrac{2t}{3(t^2-3)} = \dfrac{1}{3}$

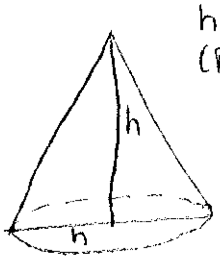
$2t = t^2 - 3$

$0 = t^2 - 2t - 3 = (t-3)(t+1)$

$t = +3$, $t = -1$ $\boxed{\begin{matrix} (0,8) \\ (8,0) \end{matrix}}$

Exercise 7. (21p220) Gravel is being dumped from a conveyor belt at a rate of 30ft^3/min and its coarseness is such that it forms a pile in the shape of a cone whose base diameter and height are always equal. How fast is the height of the pile increasing when the pile is 10ft high?

$h(t)$ height of the pile at time t (min); $V(t)$ volume of gravel at time t (Ft)



$V = \dfrac{1}{3} h \left(\dfrac{\pi h^2}{4}\right)$

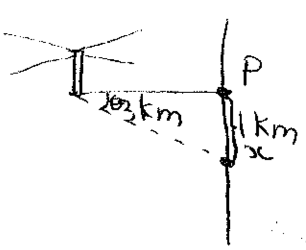
$V = \dfrac{\pi}{12} h^3$

$\dfrac{dV}{dt} = \dfrac{\pi}{12}(3h^2)\left(\dfrac{dh}{dt}\right)$

Question is: find $\dfrac{dh}{dt}$ when $\dfrac{dV}{dt} = 30$, $h = 10$

$\boxed{\dfrac{dh}{dt} = \dfrac{30 \times 4}{\pi\, 100} = \dfrac{1.2}{\pi} \approx 0.38 \text{Ft/min}}$

Exercise 8. (30p220) A lighthouse is on a small island 3km away from the nearest point P on a straight shoreline and its light makes 4 revolution per minutes. How fast is the beam of light moving along the shoreline when it is at 1km from P?



$\dfrac{d\theta}{dt} = 8\pi$ rad/min.

$\tan\theta = \dfrac{x}{3}$

$(1 + \tan^2\theta)\dfrac{d\theta}{dt} = \dfrac{1}{3}\dfrac{dx}{dt}$

$\dfrac{dx}{dt} = 3\left(1 + \dfrac{x^2}{9}\right) \times 8\pi$ ~~rad~~ $x = 1$ km/min

$= 3\left(\dfrac{10}{9}\right) \times 8\pi$ ~~rad~~ km/min

$= \dfrac{80\pi}{3} \approx 83.77$ km/min