Sections 4.3-4.4

Exercise 1. Evaluate the expression

$$A = \log_2 64 + \log_9 3 + \ln(e^{\sqrt{3}}) + \log_3(15) + \log_3(75) - 3\log_3(5) + e^{4\ln(3)}$$

$\log_2(64) = \log_2(2^6) = 6$

$\log_9(3) = \log_9(9^{1/2}) = \frac{1}{2}$

$\log_3(15) + \log_3(75) - 3\log_3(5) = \log_3(5) + \log_3(3) + 2\log_3(5) + \log_3(3) - 3\log_3(5)$
$= 2$

$e^{4\ln(3)} = e^{\ln(3^4)} = 81$

$A = 6 + 0.5 + 2 + 81 = \boxed{89.5}$

Exercise 2. Express

$$B = \log_2(x) + 5\log_2(x-2) - \frac{1}{3}\log_2(3x+4)$$

as a single logarithm.

Domain. the expression exists if $x>0$, $x-2>0$ $3x+4>0$
if $x>0$ $x>2$ $x>-\frac{4}{3}$ $\Rightarrow$ Domain $(2,\infty)$

$$B = \log_2\left(\frac{x(x-2)^5}{\sqrt[3]{3x+4}}\right)$$

Exercise 3. Solve the following equations

1. $\log_3(x) = -2$.

$3^{\log_3(x)} = x = 3^{-2} = \frac{1}{9}$ $\boxed{x = \frac{1}{9}}$

2. $\ln(2x+5) = -1$

$e^{\ln(2x+5)} = 2x+5 = e^{-1} = \frac{1}{e}$ $\boxed{x = \left(\frac{1}{e} - 5\right) \times \frac{1}{2}}$

3. $\ln(x-4) + \ln(x+2) = \ln(15)$

Domain : equation exist if $x-4>0$ and $x+2>0$
if $x>4$ and $x>-2$
if $\boxed{x>4}$

$$\ln\left((x-4)(x+2)\right) = \ln(15)$$

$$(x-4)(x+2) = 15$$

$$x^2 - 2x - 8 = 15$$

$$(x^2 - 2x + 1) = 24$$

$$(x-1)^2 = 24$$

$$x = 1 \pm \sqrt{24} = 1 \pm 2\sqrt{6}$$

$1 - 2\sqrt{6} < 4 \Rightarrow$ 1 solution >4 : $\boxed{1+2\sqrt{6}}$

4. $\log_2(\log_5(\log_3(2x+1))) = 0.$

$$\Rightarrow \log_5(\log_3(2x+1)) = 1$$

$$\log_3(2x+1) = 5$$

$$2x+1 = 3^5$$

$$\boxed{x = \frac{3^5 - 1}{2}}$$

5. $3^{2^x} = 7$

$$2^x = \log_3 7$$

$$\boxed{x = \log_2(\log_3(7))} \quad \text{or} \quad \boxed{x = \frac{\ln\left(\frac{\ln(7)}{\ln(3)}\right)}{\ln(2)}}$$

6. $5\,e^x - e^{2x} = 6.$

$$(e^x)^2 - 5(e^x) + 6 = 0$$

$$(e^x - 3)(e^x - 2) = 0$$

$$x = \ln(3) \text{ or } x = \ln(2)$$

Exercise 4. Find each limits

1. $\lim_{x\to\infty} \ln(3x^2+5) - 2\ln(x+1).$

$$\ln(3x^2+5) - 2\ln(x+1) = \ln\left(\frac{3x^2+5}{x^2+2x+1}\right)$$

$$\lim_{x\to\infty} \ln(3x^2+5) - 2\ln(x+1) = \lim_{x\to\infty} \ln\left(\frac{3x^2+5}{x^2+2x+1}\right) = \lim_{x\to\infty} \ln\left(\frac{3x^2}{x^2}\right) = \boxed{\ln(3)}$$

2. $\lim_{x\to-\infty} \ln(e^{3x}) - \ln(e^{x} - e^{2x})$.

$$\ln(e^{3x}) - \ln(e^{x} - e^{2x}) = \ln\left(\frac{e^{3x}}{e^{x} - e^{2x}}\right) = \ln\left(\frac{e^{2x}}{1 - e^{x}}\right)$$

$$\lim_{x\to-\infty} \ln(e^{3x}) - \ln(e^{x} - e^{2x}) = \lim_{x\to-\infty} \ln\left(\frac{e^{2x}}{1 - e^{x}}\right) = -\infty$$

$$\left(\lim_{x\to-\infty} \frac{e^{2x}}{1 - e^{x}} = 0\right)$$

Exercise 5. Find the inverse function of

1. $y = 2^{e^{3x}}$ $\Longleftrightarrow$ inverse function g

$$\log_2(y) = e^{3x}$$

$$\boxed{g(y) = \frac{\ln(\log_2(y))}{3} \quad \text{or} \quad g(y) = \frac{1}{3}\ln\left(\frac{\ln(y)}{\ln(2)}\right)}$$

$$\ln(\log_2(y)) = 3x$$

$$x = \frac{\ln(\log_2(y))}{3}$$

2. $y = (\ln(e^{x} + 1))^2 \quad x > 0$

$$e^{\sqrt{y}} = e^{x} + 1$$

$$e^{x} = e^{\sqrt{y}} - 1$$

$$x = \ln(e^{\sqrt{y}} - 1)$$

$$\boxed{g(y) = \ln(e^{\sqrt{y}} - 1)}$$

Exercise 6. Find f' for

1. $f(x) = \ln(4\,e^{-x} + x\,e^{-x})$.

$$f'(x) = \frac{(4e^{-x} + xe^{-x})'}{4e^{-x} + xe^{-x}} = \frac{-4e^{-x} + e^{-x} - xe^{-x}}{4e^{-x} + xe^{-x}} = \boxed{\frac{-3e^{-x} - xe^{-x}}{4e^{-x} + xe^{-x}}}$$

2. $f(x) = \ln\sqrt{\dfrac{x^2+1}{x^2-1}}$.

$$f(x) = \frac{1}{2}\ln(x^2+1) - \frac{1}{2}\ln(x^2-1)$$

$$\boxed{f'(x) = \frac{1}{2}\left(\frac{2x}{x^2+1}\right) - \frac{1}{2}\left(\frac{2x}{x^2-1}\right)}$$

3. $f(x) = \sqrt{\ln\left(\frac{x^2+1}{x^2-1}\right)}$. $= \left(\ln(x^2+1) - \ln(x^2-1)\right)^{1/2}$

$$\boxed{f'(x) = \frac{1}{2}\left(\ln(x^2+1) - \ln(x^2-1)\right)^{-1/2}\left(\frac{2x}{x^2+1} - \frac{2x}{x^2-1}\right)}$$

4. $f(x) = x^{\sin x} - (\sin x)^x$. $= e^{\sin x \ln(x)} - e^{x\ln(\sin x)}$

$$f'(x) = \left(\cos x \ln(x) + \frac{\sin x}{x}\right) x^{\sin x} - \left(\ln(\sin x) + x\frac{\cos x}{\sin x}\right)(\sin x)^x$$

or

$$= \left(\cos x \ln(x) + \frac{\sin x}{x}\right) e^{\sin x \ln(x)} - \left(\ln(\sin x) + x\frac{\cos x}{\sin x}\right) e^{x \ln(\sin x)}$$

Exercise 7. Find an equation of the tangent line to the curve

$$y = 10^x$$

at $(1, 10)$. Slope: $y'(x) = \ln(10)\, 10^x$

$y'(1) = 10 \ln(10)$

slope-point equation $y - 10 = 10\ln(10)(x-1)$

Exercise 8. Use the logarithmic differentiation to find the derivative of

1. $f(x) = \frac{\tan^5 x (x^2 - 3x + 1)^6}{\sqrt{3x-1}(2x+5)^4}$. $\ln(f) = 5\ln(\tan x) + 6\ln(x^2 - 3x + 1) - \frac{1}{2}\ln(3x-1) - 4\ln(2x+5)$

$(\ln(f(x)))' = \frac{f'(x)}{f(x)} = 5\frac{\sec^2 x}{\tan x} + 6\frac{2x-3}{x^2-3x+1} - \frac{1}{2}\left(\frac{3}{3x-1}\right) - 4\left(\frac{2}{2x+5}\right)$

$$\boxed{f'(x) = f(x)\left(\frac{5\sec^2 x}{\tan x} + \frac{12x - 18}{x^2 - 3x + 1} - \frac{3}{6x-2} - \frac{8}{2x+5}\right)}$$

2. $g(x) = \frac{x^{2/5} e^{x^2+3x} \sin^2 x}{\sqrt[3]{x}(2x-7)^3}$. $\ln(g(x)) = \frac{2}{5}\ln(x) + x^2 + 3x + 2\ln(\sin x) - \frac{1}{3}\ln(x) - 3\ln(2x-7)$

$\ln(g(x))' = \frac{g'(x)}{g(x)} = \frac{2}{5x} + 2x + 3 + \frac{2\cos x}{\sin x} - \frac{1}{3x} - 3\left(\frac{2}{2x-7}\right)$

$$\boxed{g'(x) = g(x)\left(\frac{2}{5x} + 2x + 3 + \frac{2\cos x}{\sin x} - \frac{1}{3x} - \frac{6}{2x-7}\right)}$$