Engineering Mathematics I
Math 151, Fall 2012

Texas A&M University

Review for exam 3

Exercise 1. Find $f(x)$ for

1. $f'(x) = x - \sqrt[4]{x}$, $f(1) = 1$.

$f'(x) = x - x^{1/4}$

$f(x) = \frac{x^2}{2} - \frac{x^{5/4}}{5/4} + C$

$f(1) = 1 = \frac{1}{2} - \frac{4}{5} + C$

$C = \frac{13}{10}$

$\boxed{f(x) = \frac{x^2}{2} - \frac{4}{5}x^{5/4} + \frac{13}{10}}$

2. $f'(x) = \dfrac{3+2x}{\sqrt{x}}$, $f(1) = 0$.

$f'(x) = 3(x)^{-1/2} + 2(x^{1/2})$

$f(x) = 3\frac{x^{1/2}}{1/2} + 2\frac{x^{3/2}}{3/2} + C$

$f(1) = 0 = 6 + \frac{4}{3} + C$

$C = -\frac{22}{3}$

$\boxed{f(x) = 6x^{1/2} + \frac{4}{3}x^{3/2} - \frac{22}{3}}$

3. $f''(x) = 5\sin x - 3\cos x$, $f'(0) = 1$, $f(0) = -3$.

$f'(x) = -5\cos x - 3\sin x + C$

$f'(0) = 1 \Rightarrow -5 + C = 1 \quad C = 6$

$f'(x) = -5\cos x - 3\sin x + 6$

$f(x) = -5\sin x + 3\cos x + 6x + C$

$f(0) = -3 \Rightarrow 3 + C = -3$

$\boxed{f(x) = -5\sin x + 3\cos x + 6x - 6}$

Exercise 2. Find the vector $\vec{r}(t)$ that gives the position of a particle at time t having the acceleration $\vec{a}(t) = \langle 2t, 3\rangle$, $\vec{v}(0) = \langle 1, -1\rangle$ and initial position $(1, 2)$.

$\vec{v}(t) = \langle t^2 + v_x, 3t + v_y \rangle$

$\vec{v}(0) = \langle 1, -1 \rangle \Rightarrow v_x = 1$ and $v_y = -1$

$\vec{v}(t) = \langle t^2 + 1, 3t - 1 \rangle$

$\vec{r}(t) = \langle \frac{t^3}{3} + t + r_x, \frac{3t^2}{2} - t + r_y \rangle$

$\vec{r}(0) = \langle 1, 2 \rangle \Rightarrow r_x = 1$ and $r_y = 2$

$\boxed{r(t) = \langle \frac{t^3}{3} + t + 1, \frac{3}{2}t^2 - t + 2 \rangle}$

Exercise 3. Find the point on the hyperbola $xy = 8$ that is the closest to the point $(3, 0)$. $x > 0$

distance squared $= (x-3)^2 + y^2 = (x-3)^2 + \left(\frac{8}{x}\right)^2 = (x-3)^2 + \frac{64}{x^2} = f(x)$

Find minimum for $f(x)$

$f'(x) = 2(x-3) - \frac{2 \times 64}{x^3} = 2x^4 - 6x^3 - 128 = 2(x-4)(x^3 + x^2 + 4x + 16)$

1 critical number for $x > 0$: $x = 4$

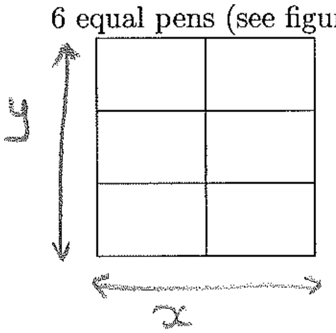
$f''(x) = 2 + \frac{3 \times 128}{x^4} > 0$ (local minimal at $x = 4$ by the 2^{nd} derivative test)

$\boxed{\text{closest point} = (4, 2)}$

Exercise 4. If $1200 cm^2$ of material is available to make a box with a square base and an open top, find the largest possible volume of the box.

See wiR 13

A farmer with 600ft of fencing wants to enclose a rectangular area and then divide it into 6 equal pens (see figure below). What is the largest possible area of the 6 pens.



$3y + 4x = 600 \text{ ft} \qquad y = 200 - \frac{4}{3}x$

area $= x \times y = x\left(200 - \frac{4}{3}x\right) = 200x - \frac{4}{3}x^2 = A(x)$

$A'(x) = 200 - \frac{8}{3}x$

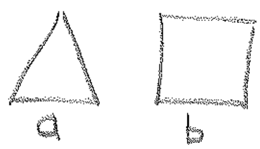
$A'(x) = 0$ if $x = \frac{600}{8} = 75 \qquad A''(x) = -\frac{8}{3} < 0$

second derivative test $\Rightarrow$ maximum

largest area $= 75\left(200 - \frac{4 \times 75}{3}\right) = 75 \times 100$

$= \boxed{7500 \text{ ft}^2}$

Exercise 5. A piece of wire of 10 inches long is cut into two pieces. One piece is bent into a square and the other is bent into an equilateral triangle. How should the wire be cut so that the total area enclosed is maximum? minimum?



$3a + 4b = 10$ inches. $b = \dfrac{10-3a}{4}$

$\text{area} = b^2 + \dfrac{a^2\sqrt{3}}{4} = \left(\dfrac{10-3a}{4}\right)^2 + \dfrac{a^2\sqrt{3}}{4} = A(a)$

$A'(a) = \dfrac{2(-3)(10-3a)}{16} + \dfrac{2a\sqrt{3}}{2} = \left(\dfrac{9}{8} + \dfrac{\sqrt{3}}{2}\right)a - \dfrac{15}{4} = 0$

$a = \dfrac{30}{9+4\sqrt{3}}$ $\quad A''(a) = \dfrac{9}{8} + \dfrac{\sqrt{3}}{2} > 0$ (local minimum)

If $a = \dfrac{30}{9+4\sqrt{3}}$, the enclosed area is minimum.

$A(0) = (2.5)^2 \approx 6.25$ $\quad A\left(\dfrac{10}{3}\right) = \dfrac{25}{3\sqrt{3}} \approx 4.8$

$\boxed{\text{area is maximum when } a = 0}$

Exercise 6. Let $f(x) = \dfrac{\ln x}{\sqrt{x}}$

1. Find the intervals on which f is increasing or decreasing?

Domain is $(0, \infty)$

$f'(x) = \dfrac{1}{x\sqrt{x}} - \dfrac{1}{2}\dfrac{\ln x}{x^{3/2}} = \dfrac{2 - \ln x}{2x^{3/2}}$

f increasing on $(0, e^2)$

f decreasing on (e^2, ∞)

$f'(x) = 0 \quad \ln x = 2 \quad x = e^2$

	$(0, e^2)$	(e^2, ∞)
$2 - \ln x$	+	−
$2x^{3/2}$	+	+
$f'(x)$	+	−
$f(x)$	↗	↘

2. Find the local minimum, local maximum values of f.

no local minimum.

local maxim at $x = e^2$

$\boxed{\text{local maximum value: } f(e^2) = \dfrac{\ln(e^2)}{\sqrt{e^2}} = \dfrac{2}{e}}$

3. Find the intervals of concavity and the inflection points.

$$f''(x) = \frac{1}{2}\left(\frac{\frac{1}{2}x^{3/2} - \frac{3}{2}x^{1/2}(2-\ln x)}{x^3}\right) = \frac{1}{4}\left(\frac{2x^{1/2} - 3x^{1/2}(2-\ln x)}{x^3}\right)$$

$$f''(x) = \frac{3\ln x - 8}{4x^{5/2}} \qquad f''(x) = 0 \text{ for } \ln x = \frac{8}{3} \quad x = e^{8/3}$$

For $x < e^{8/3}$ f is concave downward
$x > e^{8/3}$ f is concave upward

At $x = e^{8/3}$ inflection point

Exercise 7. Let $f(x) = x\sqrt{x+1}$.

1. Find the intervals on which f is increasing or decreasing? Domain $[-1, \infty)$

$$f'(x) = \sqrt{x+1} + \frac{x}{2\sqrt{x+1}} = \frac{2(x+1)+x}{2\sqrt{x+1}} = \frac{3x+2}{2\sqrt{x+1}}$$

critical number $x = -\frac{2}{3}$

	$[-1, -\frac{2}{3})$	$(-\frac{2}{3}, \infty)$
$3x+2$	$-$	$+$
$f'(x)$	$-$	$+$
f	$\searrow$	$\nearrow$

f is increasing on $(-\frac{2}{3}, \infty)$
f is decreasing on $[-1, -\frac{2}{3})$

2. Find the local minimum, local maximum values of f.

local minimum value $\frac{-2}{3\sqrt{3}}$ at $x = -\frac{2}{3}$

no local maximum

3. Find the intervals of concavity and the inflection points.

$f''(x) = \dfrac{3x+4}{4(x+1)^{3/2}}$ $\quad f''$ is never 0 in $(-1,\infty)$

$f''(x) > 0$ on $(-1,\infty)$

$\boxed{\begin{array}{l} f \text{ is concave upward on } (-1,\infty) \\ \text{no inflection point} \end{array}}$

Exercise 8. Let $f(x) = 2x + \cot x$, $0 < x < \pi$.

1. Find the intervals on which f is increasing or decreasing?

$f'(x) = 2 - \csc^2 x$

$f'(x) = 0$ when $\csc^2 x = 2$ $\quad x = \frac{\pi}{4}$ $\quad x = \frac{3\pi}{4}$

	$\left(0,\frac{\pi}{4}\right)$	$\left(\frac{\pi}{4},\frac{3\pi}{4}\right)$	$\left(\frac{3\pi}{4},\pi\right)$
$f'(x)$	$-$	$+$	$-$
f	$\searrow$	$\nearrow$	$\searrow$

$\boxed{\begin{array}{l} f \text{ decreasing on } \left(0,\frac{\pi}{4}\right) \text{ on } \left(\frac{3\pi}{4},\pi\right) \\ f \text{ increasing on } \left(\frac{\pi}{4},\frac{3\pi}{4}\right) \end{array}}$

2. Find the local minimum, local maximum values of f.

$\boxed{\begin{array}{l} \text{local minimum value } f\left(\frac{\pi}{4}\right) = \frac{\pi}{2} + 1 \\ \text{local maximum value } f\left(\frac{3\pi}{4}\right) = \frac{3\pi}{2} - 1 \end{array}}$

3. Find the intervals of concavity and the inflection points.

$f''(x) = 2\cot x \sec^2 x$

$f''(x) = 0$ for $x = \frac{\pi}{2}$,

	$(0, \frac{\pi}{2})$	$(\frac{\pi}{2}, \pi)$
f''	+	−
f	concave up	concave down

$\boxed{\begin{array}{l} f \text{ concave upward for } x \in (0, \frac{\pi}{2}) \\ f \text{ concave downward for } x \in (\frac{\pi}{2}, \pi) \end{array}}$

Exercise 9. $f(x) = \sqrt{x^2 + 1} - x$.

1. Find the asymptotes.

$\lim_{x \to -\infty} f(x) = +\infty$ no asymptote at $-\infty$

$\lim_{x \to +\infty} f(x) = \lim_{x \to \infty} \frac{x^2 + 1 - x^2}{\sqrt{x^2+1} + x^2} = 0$

$\boxed{y = 0 \text{ horizontal asymptote at } +\infty}$

2. Find the intervals on which f is increasing or decreasing?

$f'(x) = \frac{2x}{2\sqrt{x^2+1}} - 1 = \frac{x - \sqrt{x^2+1}}{\sqrt{x^2+1}}$

$f'(x) = 0$ if $x = \sqrt{x^2+1}$ $x^2 = x^2 + 1$ impossible

$f'(x) < 0$ always.

f decreasing on $\mathbb{R}$

3. Find the intervals of concavity and the inflection points.

$$f''(x) = \frac{1}{(x^2+1)^{3/2}} > 0 \quad \text{for all } x$$

f is concave upward on $\mathbb{R}$

Exercise 10. $f(x) = x^5(x-1)^6$

1. Find the intervals on which f is increasing or decreasing?

$f'(x) = x^4(x-1)^5(11x-5)$

critical numbers $x=0 \quad x=1 \quad x=\frac{5}{11}$

	$(-\infty, 0)$	$(0, \frac{5}{11})$	$(\frac{5}{11}, 1)$	$(1, \infty)$
x^4	+	+	+	+
$(x-1)^5$	−	−	−	+
$(11x-5)$	−	−	+	+
f'	+	+	−	+

f increasing on $(0, \frac{5}{11})$ and on $(1, \infty)$

f decreasing on $(\frac{5}{11}, 1)$

2. Find the local minimum, local maximum values of f.

local maximum value $= \left(\frac{5}{11}\right)^5\left(\frac{6}{11}\right)^6$ at $x = \frac{5}{11}$

local minimal value 0 at $x=1$

3. Find the intervals of concavity and the inflection points.

$f''(x) = 10(x-1)^4 x^3 (11x^2 - 10x + 2) = 0$ for $x = 0$, $x = 1$

$x = \frac{5-\sqrt{3}}{11}$, $x = \frac{5+\sqrt{3}}{11}$

	$(-\infty,0)$	$\left(0,\frac{5-\sqrt{3}}{11}\right)$	$\left(\frac{5-\sqrt{3}}{11},\frac{5+\sqrt{3}}{11}\right)$	$\left(\frac{5+\sqrt{3}}{11},1\right)$	$(1,\infty)$
x^3	$-$	$+$	$+$	$+$	$+$
$11x^2-10x+2$	$+$	$+$	$-$	$+$	$+$
$f''(x)$	$-$	$+$	$-$	$+$	$+$

f concave upward on $\left(0,\frac{5-\sqrt{3}}{11}\right)$ $\left(\frac{5+\sqrt{3}}{11},\infty\right)$ f concave downward on $(-\infty,0)$ and $\left(\frac{5-\sqrt{3}}{11},\frac{5+\sqrt{3}}{11}\right)$

$\boxed{\text{inflection points at } x=0,\ x=\frac{5-\sqrt{3}}{11},\ x=\frac{5+\sqrt{3}}{11}}$

Exercise 11. Find the absolute maximum and absolute minimum values of f on the given interval.

1. $f(x) = \dfrac{x}{x+1}$, $[1,2]$

$f'(x) = \frac{(x+1)-x}{(x+1)^2} = \frac{1}{(x+1)^2} > 0$ on $[1,2]$

no critical numbers, f increasing on $[1,2]$

$\boxed{\begin{array}{l}\text{absolute maximum value } f(2) = \frac{2}{3} \\ \text{absolute minimum value } f(1) = \frac{1}{2}\end{array}}$

2. $f(x) = \sqrt{9-x^2}$, $[-1,2]$.

$f'(x) = \frac{-2x}{2\sqrt{9-x^2}}$ $\qquad f'(x) = 0$ when $x = 0$

$f(-1) = \sqrt{8}$

$f(2) = \sqrt{9-4} = \sqrt{5}$

$f(0) = 3$

$\boxed{\begin{array}{l}\text{absolute maximum value } 3 \text{ at } x=0 \\ \text{absolute minimum value } \sqrt{5} \text{ at } x=2\end{array}}$

3. $f(x) = x^2 - 2x + 2$, $[0, 3]$.

$f'(x) = 2x - 2$ critical numbers : $x = 1$

$f(0) = 2$
$f(3) = 9 - 6 + 2 = 5$
$f(1) = 1 - 2 + 2 = 1$

absolute maximum value 5
absolute minimum value 1

4. $f(x) = \dfrac{\cos x}{2 + \sin x}$, $[0, 2\pi]$.

$f'(x) = -\dfrac{2\sin x + 1}{(2 + \sin x)^2}$ $f'(x) = 0$ if $\sin x = \frac{1}{2}$ $x = \frac{11\pi}{6}$, $x = \frac{7\pi}{6}$

$f(0) = \frac{1}{2}$ $f\left(\frac{7\pi}{6}\right) = -\frac{1}{\sqrt{3}}$
$f(2\pi) = \frac{1}{2}$
$f\left(\frac{11\pi}{6}\right) = \frac{1}{\sqrt{3}}$

absolute maximum value $\frac{1}{\sqrt{3}}$
absolute minimum value $-\frac{1}{\sqrt{3}}$

Exercise 12. Find the critical numbers of $f(x) = |x^2 - 1|$.

$f(x) = x^2 - 1$ For $1 < x$ and for $x < -1$

$f(x) = 1 - x^2$ For $-1 < x < 1$

$f'(x) = 2x$ For $1 < x$, $x < -1$ $\Rightarrow$ no critical numbers in those intervals.

$f'(x) = -2x$ For $-1 < x < 1$

$\Rightarrow$ 1 critical number at $x = 0$

at $x = 1$ $f'_L(1) = -2 \neq f'_R(1) = 2$ $\Rightarrow$ $f'(1)$ DNE.
derivative from the left $\neq$ derivative from the right
similarly $f'(-1)$ DNE

critical numbers $-1, 1, 0$

Exercise 13. Find the critical numbers of $f(x) = \sqrt[3]{x^2 - x}$.

$$f'(x) = \frac{1(2x-1)}{3(x^2-x)^{2/3}}$$

* critical number when $f'(x)$ DNE

$x = 0$ or $x = 1$

* critical number when $f'(x) = 0$

$x = \frac{1}{2}$

$$\boxed{0, \tfrac{1}{2}, 1}$$

Exercise 14. Find the following limits

1. $\lim_{x \to 0} \frac{6^x - 5^x}{x}$.

"$\frac{0}{0}$" $= \lim_{x \to 0} \frac{\ln(6)\, 6^x - \ln(5)\, 5^x}{1} = \boxed{\ln(6) - \ln(5)}$

$= \boxed{\ln\left(\frac{6}{5}\right)}$

2. $\lim_{x \to 0} \frac{\tan x - \sin x}{x^3}$.

"$\frac{0}{0}$" $= \lim_{x \to 0} \frac{\sec^2 x - \cos x}{3x^2}$ "$\frac{0}{0}$" $= \lim_{x \to 0} \frac{2\sec^2 x \tan x - \sin x}{6x}$

"$\frac{0}{0}$" $= \lim_{x \to 0} \frac{\frac{2\sin x}{\cos^3 x} - \sin x}{6x} = \lim_{x \to 0} \frac{\sin x\,(2 - \cos^3 x)}{\cos^3 x\; 6x}$

$= \lim_{x \to 0} \frac{\sin x}{x} \times \left(\frac{2 - \cos^3 x}{6\cos^3 x}\right)$

$= 1 \times \frac{1}{6} = \boxed{\frac{1}{6}}$

3. $\lim_{x\to 0}\dfrac{\sin x}{e^x}$. $= \dfrac{0}{1} = \boxed{0}$

4. $\lim_{x\to 0}\left(\dfrac{2x+\operatorname{Arcsin} x}{3x-\operatorname{Arctan} x}\right)$. $\overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x\to 0} \dfrac{2+\frac{1}{\sqrt{1-x^2}}}{3-\frac{1}{1+x^2}} = \boxed{\dfrac{3}{2}}$

5. $\lim_{x\to\infty} e^{-x}\ln x$. $= \lim_{x\to\infty}\dfrac{\ln x}{e^x} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{x\to\infty}\dfrac{\frac{1}{x}}{e^x} = \lim_{x\to\infty}\dfrac{1}{xe^x} = \boxed{0}$

6. $\lim_{x\to 0}\sqrt{x}\ln(x)$. $= \lim_{x\to 0} \frac{\ln x}{x^{-1/2}}$ "$\frac{-\infty}{+\infty}$" $= \lim_{x\to 0} \frac{\frac{1}{x}}{-\frac{1}{2}(x^{-3/2})} = \lim_{x\to 0} -2\sqrt{x} = 0$

7. $\lim_{x\to 1}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right)$. "$\frac{0}{0}$" $= \lim_{x\to 1} \frac{x-1-\ln(x)}{\ln(x)(x-1)} = \lim_{x\to 1} \frac{1-\frac{1}{x}}{\frac{(x-1)}{x} + \ln x}$ "$\frac{0}{0}$"

$= \lim_{x\to 1} \frac{x-1}{x-1+x\ln x} = \lim_{x\to 1} \frac{1}{1+\ln x + 1} = \boxed{\frac{1}{2}}$

8. $\lim_{x\to 0}(\sin x)^{\tan x}$.

$\ln\left((\sin x)^{\tan x}\right) = \tan x \ln(\sin x) = \frac{\ln(\sin x)}{\cot x}$

$\lim_{x\to 0} \frac{\ln(\sin x)}{\cot x} = \lim_{x\to 0} \frac{\frac{\cos x}{\sin x}}{-\csc^2 x} = \lim_{x\to 0} -\frac{\cos x}{\sin x} \sin^2 x = 0$

$\lim_{x\to 0} (\sin x)^{\tan x} = e^0 = \boxed{1}$

9. $\lim\limits_{x\to\infty}\left(1+\dfrac{3}{x}\right)^{5x}$.

$$\ln(y) = 5x\ln\left(1+\frac{3}{x}\right) = 5\,\frac{\ln\left(1+\frac{3}{x}\right)}{\frac{1}{x}}$$

$$\lim_{x\to\infty}\ln(y) = 5\lim_{x\to\infty}\frac{\frac{\frac{-3}{x^2}}{1+\frac{3}{x}}}{\frac{-1}{x^2}} = \lim_{x\to\infty} 5\,\frac{\frac{-3}{1+\frac{3}{x}}}{-1} = 15$$

$$\boxed{\lim_{x\to\infty} y = e^{15}}$$

10. $\lim\limits_{x\to\infty}(\mathrm{e}^x+x)^{1/x}$.

$$\ln(y) = \frac{e^x+x}{x}$$

$$\lim_{x\to\infty}\ln(y) = \lim_{x\to\infty}\frac{e^x+1}{1} = +\infty$$

$$\lim y = +\infty$$

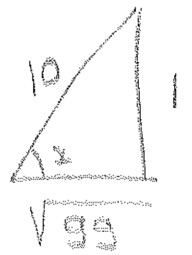
Exercise 15. Simplify the expressions

1. $\operatorname{Arctan}\left(\tan\left(\frac{6\pi}{5}\right)\right)$

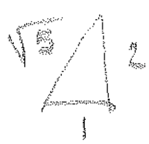
$$\tan\left(\frac{6\pi}{5}\right) = \tan\left(\frac{\pi}{5}\right)$$

$$\operatorname{Arctan}\left(\tan\left(\frac{6\pi}{5}\right)\right) = \frac{\pi}{5}$$

2. cos(Arcsin 0.1). $= \dfrac{\sqrt{99}}{10}$



3. sec(Arctan 2). $= \sqrt{5}$



4. tan(Arccos 0.5) $= \tan\left(\dfrac{\pi}{3}\right) = \sqrt{3}$

Exercise 16. Find the derivative of $f(x) = \text{Arctan}(\sin x)$ and simplify.

$$f'(x) = \frac{\cos x}{1 + \sin^2 x}$$

Exercise 17. Find the domain and the derivative of $f(x) = \text{Arccos}(\text{Arcsin}\, x)$.

domain of Arccos x is $[-1, 1]$.

$f(x)$ exists if $\text{Arcsin}\, x \in [-1, 1]$

if $\boxed{x \in [\text{Arcsin}(-1), \text{Arcsin}(1)]}$

$$f'(x) = \frac{1}{\sqrt{1-x^2}} \cdot \frac{-1}{\sqrt{1-\text{Arcsin}^2 x}}$$

Exercise 18. Find the domain and the derivative of $\text{Arccos}(e^x)$.

$-1 \leq e^x \leq 1$

$\Rightarrow x \leq 0$

domain $(-\infty, 0]$

$$f'(x) = \frac{-e^x}{\sqrt{1-(e^x)^2}}$$

Exercise 19. Find the domain and the derivative of $\text{Arcsin}((1 - x^2)$.

domain $-1 \leq 1 - x^2 \leq 1$

$-2 \leq -x^2 \leq 0$

$\boxed{-\sqrt{2} \leq x \leq \sqrt{2}}$

$$f'(x) = \frac{-2x}{\sqrt{1-(1-x^2)^2}}$$

Exercise 20. Find $\lim_{x\to 0^+} \text{Arctan}(\ln 3x)$.

$\lim \ln(3x) = -\infty$

$\lim_{x\to 0^+} \text{Arctan}(\ln(3x)) = -\frac{\pi}{2}$

Exercise 21. Find $\lim_{x\to\infty} \text{Arccos}\left(\frac{2x-1}{2x+1}\right)$.

$\frac{2x-1}{2x+1} \xrightarrow[x\to\infty]{} 1 \Rightarrow \lim_{x\to\infty} \text{Arccos}\left(\frac{2x-1}{2x+1}\right) = \text{Arccos}(1) = 0$

Exercise 22. Find $\lim_{x\to\infty} \text{Arcsin}\left(\frac{x-1}{x+1}\right)$.

$\frac{x-1}{x+1} \xrightarrow[x\to\infty]{} 1 \Rightarrow \lim_{x\to\infty} \text{Arcsin}\left(\frac{x-1}{x+1}\right) = \text{Arcsin}(1) = \frac{\pi}{2}$

Exercise 23. Polonium-210 has half life of 140 days.

1. If a sample has a mass of 200mg, find a formula for the mass that remains after t days.

2. Find the mass after 100 days.

3. When will the mass be reduced to 10mg?

Exercise 24. A bacteria culture starts with 400 bacteria and after 3 hours there are 3200 bacteria.

1. Find an expression for the number of bacteria after t hours.

2. Find the number of bacteria after 4 hours.

3. When will the number of bacteria reach 10,000?

Exercise 25. A roast turkey is taken from an oven when its temperature has reached 185°F and is placed on a table in a room where the temperature is 75°F.

1. If the temperature of the turkey is 150°F after half an hour, what is the temperature after 45 min?

2. When will the turkey have cooled to 100°?

Exercise 26. Evaluate

1. $\log_2(64)$, $\log_8(32)$.

$$\log_2(64) = \log_2(2^6) = 6$$
$$\log_8(32) = \log_8(8^{5/3}) = \frac{5}{3}$$

2. $\log_3(9^{\sqrt{3}})$ $= \log_3\left(3^{2\sqrt{3}}\right) = 2\sqrt{3}$

3. $e^{\ln(2)} + \ln(e^{\sqrt{2}})$ $= 2 + \sqrt{2}$

Exercise 27. Find the domain and solve the equations

1. $\ln(2e^x - 1) = 3$

domain $2e^x - 1 > 0$ $e^x > \frac{1}{2}$ $x > \ln\left(\frac{1}{2}\right) = -\ln(2)$

$[-\ln(2), \infty)$

$2e^x - 1 = e^3$

$e^x = \frac{e^3 + 1}{2}$ $\boxed{x = \ln\left(\frac{e^3+1}{2}\right)}$

2. $\ln(x) + \ln(x-1) = \ln(.5)$

Domain $x > 0$ and $x - 1 > 0$

$\Rightarrow x > 0$ $(0, \infty)$

$\ln(x(x-1)) = \ln\left(\frac{1}{2}\right)$

$x^2 - x - \frac{1}{2} = 0$

$\left(x - \frac{1}{2}\right)^2 - \frac{3}{4} = 0$

$x = \frac{1}{2} + \frac{\sqrt{3}}{2}$ (second solution not in the domain)

3. $\log_2(\log_3(\log_4(x))) = 0$

domain : exists if $\log_3(\log_4(x)) > 0$

$\Rightarrow \log_4(x) > 1$

$\boxed{x > 4}$

solve

$\log_3(\log_4(x)) = 1$

$\Rightarrow \log_4(x) = 3 \Rightarrow \boxed{x = 4^3}$

4. $\ln\left(\dfrac{x-2}{x-1}\right) = 1 + \ln\left(\dfrac{x-3}{x-1}\right)$

domain $\left(\frac{x-2}{x-1}\right) > 0$ and $\left(\frac{x-3}{x-1}\right) > 0$ $\left(\begin{matrix} x > 2 \\ \text{or } x < 1 \end{matrix}\right)$ or $\left(\begin{matrix} x > 3 \\ \text{or } x < 1 \end{matrix}\right)$

$\Rightarrow \boxed{(-\infty, 1) \cup (3, \infty)}$

$\ln\left(\frac{x-2}{x-1}\right) - \ln\left(\frac{x-3}{x-1}\right) = 1 \Rightarrow \ln\left(\frac{x-2}{x-3}\right) = 1 \Rightarrow \frac{x-2}{x-3} = e$

$\Rightarrow x - 2 = e(x-3) \Rightarrow x(1-e) = 2 - 3e \Rightarrow \boxed{x = \left(\frac{2-3e}{1-e}\right)}$

Exercise 28. Find the derivative of

1. $f(x) = \sqrt{x}\ln(x)$ and state the domain.

domain $(0, \infty)$

$\boxed{f'(x) = \frac{1}{2\sqrt{x}}\ln x + \frac{\sqrt{x}}{x}}$

2. $g(x) = \ln\left(\frac{x^2 \sin x}{(\text{Arctan}\, x)\sqrt{x^2+3}}\right)$ $= 2\ln(x) + \ln(\sin x) - \ln(\text{Arctan}\, x) - \frac{1}{2}\ln(x^2+3)$

$$\boxed{g'(x) = \frac{2}{x} + \frac{\cos x}{\sin x} - \frac{1}{(1+x^2)\,\text{Arctan}\, x} - \frac{1}{2}\left(\frac{2x}{x^2+3}\right)}$$

3. $f(x) = \ln(\sec x + \tan x)$ and simplify.

$$f'(x) = \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} = \sec x$$

4. $f(x) = x^{\ln(x)} - (\ln x)^x$.

$\ln(y) = \ln(x^{\ln x}) = (\ln(x))^2$

$\frac{y'}{y} = 2\left(\frac{1}{x}\right)\ln(x)$

$y' = \frac{2}{x}\ln(x)\, x^{\ln x}$

$\ln z = \ln(\ln x)^x$

$= x \ln(\ln x)$

$\frac{z'}{z} = \ln(\ln x) + x\,\frac{\frac{1}{x}}{\ln(x)}$

$z' = \left(\ln(\ln(x)) + \frac{1}{\ln x}\right)\ln(x)^x$

$$\boxed{f'(x) = \frac{2\ln(x)}{x}\, x^{\ln x} + \left(\ln(\ln(x)) + \frac{1}{\ln(x)}\right)\ln(x)^x}$$

5. $h(x) = (\sin x)^{\cos x}$

$$\ln(h(x)) = \cos x \ \ln(\sin x)$$

$$\frac{h'(x)}{h(x)} = \left(-\sin x \ \ln(\sin x) + \frac{\cos^2 x}{\sin x}\right)$$

$$h'(x) = \left(-\sin x \ \ln(\sin x) + \frac{\cos^2 x}{\sin x}\right)(\sin x)^{\cos x}$$

Exercise 29. Using the logarithmic differentiation, find the derivative of

$$f(x) = \sqrt{\frac{(\cos^6 x)(e^{3x^2})\sqrt{x^5 - 1}}{(2x+1)x^{2/5}}}$$

$$\ln(f(x)) = \frac{1}{2}\Big(6\ln(\cos x) + \underbrace{\ln(e^{3x^2})}_{3x^2} + \frac{1}{2}\ln(x^5-1) - \ln(2x+1) - \frac{2}{5}\ln(x)$$

$$\frac{f'(x)}{f(x)} = \frac{1}{2}\left(-\frac{6\sin x}{\cos x} + 6x + \frac{1}{2}\cdot\frac{5x^4}{x^5-1} - \frac{2}{2x+1} - \frac{2}{5x}\right)$$

$$f'(x) = \frac{1}{2}\left(-6\tan x + 6x + \frac{5}{2}\frac{x^4}{x^5-1} - \frac{2}{2x+1} - \frac{2}{5x}\right)f(x)$$

Exercise 30. Find y' if $y = \ln(x^2 + y^2)$.

$$\frac{dy}{dx} = \frac{2x + 2y\frac{dy}{dx}}{x^2+y^2}$$

$$\frac{2x}{x^2+y^2} = \left(1 - \frac{2y}{x^2+y^2}\right)\frac{dy}{dx}$$

$$\boxed{\frac{dy}{dx} = \left(\frac{2x}{x^2+y^2-2y}\right)}$$