Section 4.3,4,4 Logarithmic functions

Definition & properties:

$$\log_a(x) = y \quad \Leftrightarrow a^y = x$$

$$\log_a(a^x) = x, \quad x \in \mathbb{R}. \qquad a^{\log_a(x)} = x, \quad x > 0.$$

$$\log_a(xy) = \log_a(x) + \log_a(y)$$

$$\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$$

$$\log_a(x^y) = y\log_a(x)$$

Exercise 1. Evaluate the expressions

$$\log_3(3^{\sqrt{7}}) + \log_3(\frac{1}{27})$$

$$\log_8(6) - \log_8(3) + \log_8(4)$$

$$\log_a(a)$$

Definition (natural logarithm):

$$\log_e(x) = ln(x)$$

$$\ln(x) = y \quad \Leftrightarrow e^y = x$$

$$\ln(e^x) = x, \quad x \in \mathbb{R}. \qquad e^{\ln(x)} = x, \quad x > 0$$

$$ln(\mathrm{e}) = 1, \quad \ln(1) = 0$$

$$\log_a(x) = \frac{\ln(x)}{\ln(a)}, \quad a^x = \mathrm{e}^{x\ln(a)}$$

$$\lim_{x\to 0}\ln(x) = -\infty, \qquad \lim_{x\to\infty}\ln(x) = \infty.$$

Exercise 2. Solve the following equations

$$\ln(x+3) + \ln(x-1) = \ln(5)$$

$$2\ln(x) = \ln(x^2 + 3x + 5)$$

$$\ln\left(\frac{x-2}{x-1}\right) = 1 + \ln\left(\frac{x-3}{x-1}\right)$$

Exercise 3. Find the following limits

$$\lim_{x\to 6^+}\ln(x-6)$$

$$\lim_{x\to\infty} 3\ln(x-1) - \ln(2x^3 - 3x + 5)$$

Exercise 4. Find the inverse of the function

$$y = \ln(1 + \mathrm{e}^{2x})$$

Theorem:

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln x = \frac{1}{x}, \qquad \frac{\mathrm{d}}{\mathrm{d}x}\ln(g(x)) = \frac{g'(x)}{g(x)}.$$

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$$\frac{\mathrm{d}}{\mathrm{d}x}\log_a(x) = \frac{1}{x\ln(a)}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}a^x = a^x\ln(a)$$

Exercise 5. Find the derivative of

$$f(x) = \ln(\sec x + \tan x)$$

$$g(x) = \ln\sqrt{\frac{3x+2}{x-5}}$$

$$h(x) = \frac{1 - 2\ln x}{3 + \ln u}$$

$$j(x) = (\sin x)^x$$

$$k(x) = x\ln(x) - x$$

Exercise 6. Use the logarithmic differentiation to find th derivative of

$$f(x) = \frac{(x+3)^7(x-1)^3}{\sqrt{x-2}(2x-5)^2}$$

Exercise 7. Find a quadratic approximation of $\ln(x)$ near 1.

Exercise 8. If $f(x) = \mathrm{e}^x + \ln(x)$ and $g(x) = f^{-1}(x)$, find $g'(\mathrm{e})$.

Exercise 9. Given a positive real number a, find the limit

$$\lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^x$$