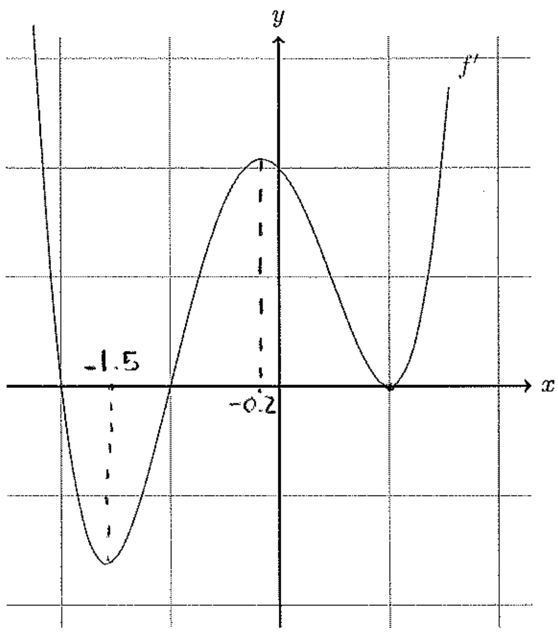
Sections 5.1, 5.2, 5.3



Exercise 1. Given the graph of f' below,

1. Determine over which interval is f increasing? decreasing?

f is increasing when $f'(x)>0$. ie when $-1<x<\infty$ and $x<-2$

f is decreasing when $f'(x)<0$ i.e. when $-2<x<-1$

2. What are the critical numbers of f? Does the function f has local maxima, local minima at the critical numbers.

x is a critical number of f if $f'(x)$ doesnot exist or $f'(x)=0$. ie $x=-2$, $x=-1$, $x=1$

3 critical numbers $x=-2$, $x=-1$, $x=1$.

3. Determine over which interval is f concave upward? concave downward? Find the inflection points.

f is concave up if $f''>0$ or $f'\nearrow$. when $x\in(-1.5,-0.2)$ and $x\in(1,\infty)$

f is concave down if $f''<0$ or $f'\searrow$. when $x\,(-\infty,-1.5)$, $(-0.2,1)$

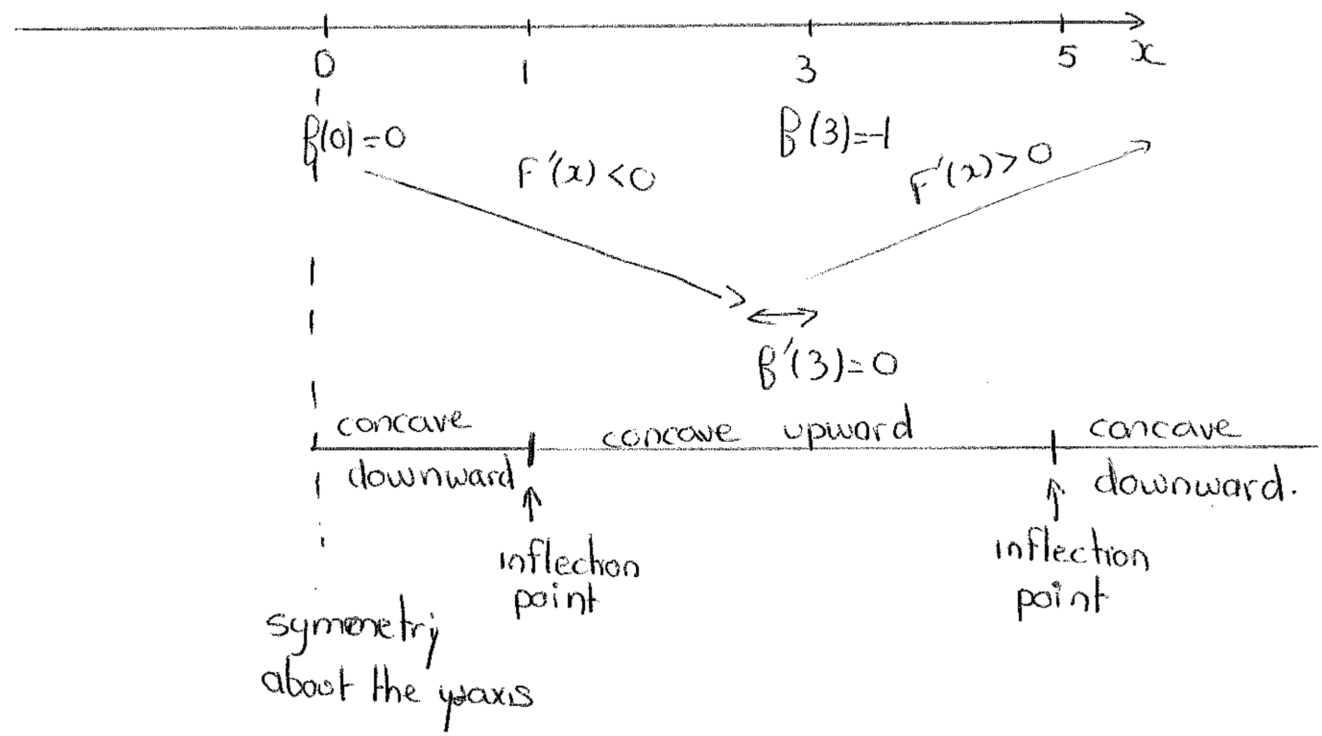
concave up: $(-1.5,-0.2)$ $(1,\infty)$

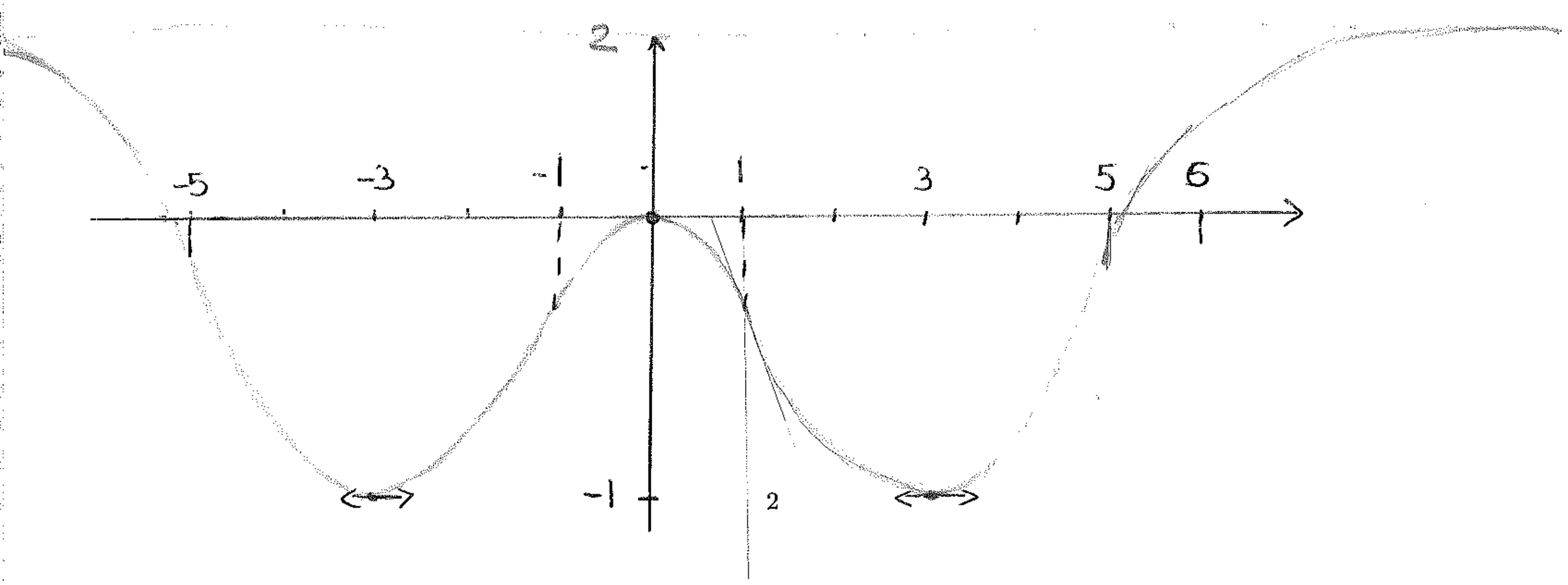
concave down $(-\infty,-1.5)$ $(-0.2,1)$

Exercise 2. Sketch a graph of a continuous function f satisfying the following properties:

1. $f(0) = 0$, $f(3) = -1$, $f'(3) = 0$.
2. $f'(x) < 0$ if $0 < x < 3$; $f'(x) > 0$ if $x > 3$.
3. $f''(x) < 0$ if $x < 1$ and $x > 5$; $f''(x) > 0$ if $1 < x < 5$.
4. $\lim_{x\to\infty} f(x) = 2$.
5. $f(-x) = f(x)$.

1.





Exercise 3. On which intervals is f increasing? Find the local extrema.

1. $f(x) = x^3 + 3x^2 - 9x + 5$.

f is increasing if $f'(x) > 0$

$f'(x) = 3x^2 + 6x - 9 = 3(x^2 + 2x - 3) = 3(x+3)(x-1)$

x	$x+3$	$x-1$	$f'(x)$
$(-\infty, -3)$	$-$	$-$	$+$ $\nearrow$
$(-1 \quad 3)$	$+$	$-$	$-$ $\searrow$
$(3 \quad \infty)$	$+$	$+$	$+$ $\nearrow$

f is increasing on $(-\infty \; -3) \quad (1, \infty)$

local maximum at -3 local maximum value : $f(-3) = 32$

local minimum at 1 local minimum value $f(1) = 0$

2. $f(x) = 2x^3 - 3x^2 + 5$.

$f'(x) = 6x^2 - 6x = 6(x(x-1))$

f is increasing on $(-\infty, 0]$ and on $[1, \infty)$

x	$(-\infty \quad 0)$	0	$(0, 1)$	1	$(1 \; +\infty)$
$f'(x)$	$+$	0	$-$	0	$+$
f	$\nearrow$		$\searrow$		$\nearrow$

local maximum value $= 5$ at $x = 0$

local minimum value $= 4$ at $x = 1$

3. $f(x) = \frac{(x-2)^2}{x^2+1}$ Domain $= \mathbb{R}$

$$F'(x) = \frac{2(x-2)(x^2+1) - 2x(x-2)^2}{(x^2+1)^2} = \frac{2(x-2)\left(x^2+1-x(x-2)\right)}{(x^2+1)^2}$$

$$= \frac{2(x-2)(2x+1)}{(x^2+1)^2}$$

x	$(-\infty, -\frac{1}{2})$	$-\frac{1}{2}$	$(-\frac{1}{2}, 2)$	2	$(2\ \infty)$
$x-2$	$-$		$-$	0	$+$
$2x+1$	$-$	0	$+$		$+$
$F'(x)$	$+$	0	$-$	0	$+$
F	$\nearrow$		$\searrow$		$\nearrow$

* F is increasing on $(-\infty, -\frac{1}{2})$ and on $(2, \infty)$
* local maximum $F(-\frac{1}{2}) = 5$ at $x = -\frac{1}{2}$
* local minimum $F(2) = 0$ at $x = 2$

Exercise 4. Find the critical numbers for the following functions.

1. $f(x) = x^3 + 3x^2 - 9x + 5$

x is a critical number if $f'(x) = 0$

$f'(x) = 3x^2 + 6x - 9 = 3(x+3)(x-1)$

2 critical numbers : $x = -3$ and $x = +1$

2. $f(x) = |x^2 - 3x + 2|$

critical number when $f'(x) = 0$ or $f'(x)$ DNE

$x^2 - 3x + 2 = (x-2)(x-1)$

If $x < 1$ or $x > 2$ $\quad f(x) = x^2 - 3x + 2$.

If $1 < x < 2$, $\quad f(x) = -(x^2 - 3x + 2)$

At $x = 1$ right slope $f'_+(1) = -(2-3) = 1$
left slope $f'_-(1) = 2 - 3 = -1$ $\Rightarrow$ $f'(1)$ DNE
and 1 is a critical number

At $x = 2$ right slope $f'_+(2) = 4 - 3 = 1$
leftslope $f'_-(2) = -4 + 3 = -1$ $\quad f'(2)$ DNE
and 2 is a critical number

If $g(x) = x^2 - 3x + 2$ $\quad f(x) = \pm g(x)$ depending on x

$g'(x) = 2x - 3$

$g'(x) = 0$ if $2x - 3 = 0$ $\quad x = \frac{3}{2}$ $\quad x = \frac{3}{2}$ is a critical number for g.

$x = \frac{3}{2} \in (1,2)$ $\quad f(x) = -g(x)$ on that interval.

$f'(x) = -g'(x)$

$\Rightarrow \frac{3}{2}$ is a critical number for f too.

$$\boxed{\begin{array}{lll} \text{3 critical numbers} & x = 1 \text{ and } x = 2 & (\text{where } f'(x) \text{ DNE}) \\ & x = \frac{3}{2} & (\text{where } f'(x) = 0) \end{array}}$$

3. $f(x) = x^2 \ln(x)$ Domain $(0, \infty)$

$F'(x) = 2x \ln(x) + \frac{x^2}{x} = x(2\ln(x) + 1)$

$F'(x) = 0$ if $\ln(x) = -\frac{1}{2}$ if $x = e^{-\frac{1}{2}}$

1 critical point $x = e^{-1/2}$

4. $f(x) = x^{(2/3)}(2x - 5)$

Domain $= \mathbb{R}$

$f'(x) = \frac{2}{3} x^{(-1/3)} (2x - 5) + 2(x^{2/3})$

$= x^{-\frac{1}{3}} \left(\frac{4}{3}x - \frac{10}{3} + 2x \right)$

$= x^{-1/3} \left(\frac{10x - 10}{3} \right)$

2 critical numbers : $x = 0$ ($f'(x)$ DNE)

$x = 1$ ($f'(x) = 0$)

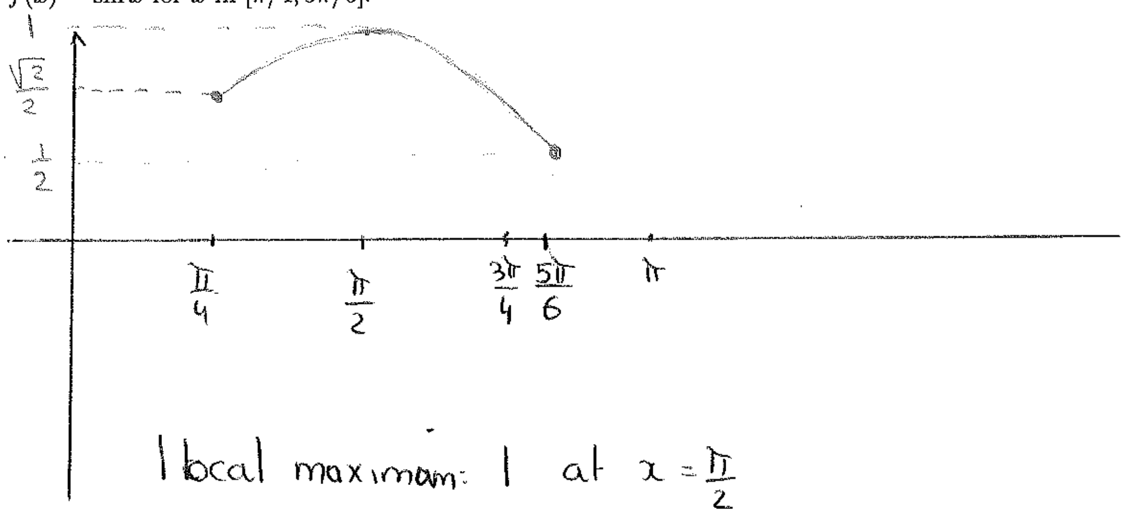
5. $f(x) = x^2 e^{-x^2}$ Domain $= \mathbb{R}$

$$f'(x) = 2x\,e^{-x^2} - 2x^3 e^{-x^2}$$
$$= 2x(1-x^2)\,e^{-x^2}$$
$$= 2x(1-x)(1+x)\,e^{-x^2}$$

3 critical numbers $x=-1,\ x=0,\ x=1$

Exercise 5. Find all absolute and local extrema for the following functions by sketching the graph.

1. $f(x) = \sin x$ for x in $[\pi/4, 5\pi/6]$.



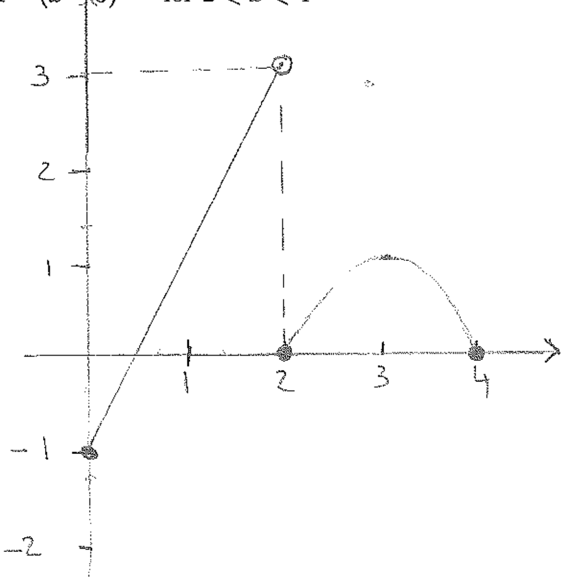
1 local maximum: 1 at $x = \frac{\pi}{2}$

0 local minimum.

Absolute maximum 1 at $x = \frac{\pi}{2}$

Absolute minimum $\frac{1}{2}$ at $x = \frac{5\pi}{6}$

2. $f(x) = \begin{cases} 2x-1, & \text{for } 0 \leqslant x < 2 \\ 1-(x-3)^2 & \text{for } 2 \leqslant x \leqslant 4 \end{cases}$



1 local maximum at $x=3$ local maximum value $=1$

No local minimum.

1 absolute minimum value $= -1$

No absolute maximum value

Exercise 6. Find the absolute maximum and absolute minimum values for the following functions on the given interval.

1. $f(x) = \frac{x^2}{x+4}$ for x in $[-2, 6]$.

critical points: $F'(x) = \dfrac{2x(x+4) - x^2}{(x+4)^2} = \dfrac{x^2 + 8x}{(x+4)^2}$

1 critical number in $[-2, 6]$ $= 0$

$F(0) = 0$
$F(-2) = 2$
$F(6) = 3.6$
absolute maximum value 3.6 at $x = 6$
absolute minimum value 0 at $x = 0$

2. $f(x) = 2\sin x + (\cos^2 x)$, for x in $[0, 2\pi]$

critical numbers when $f'(x) = 0$
$F'(x) = 2\cos x - 2\cos x \sin x$
$= 2\cos x(1 - \sin x)$
2 critical numbers $x = \frac{\pi}{2}$, $x = \frac{3\pi}{2}$

$F(0) = 1$

$F(\frac{\pi}{2}) = 3$

$F(\frac{3\pi}{2}) = -2 + 1 = -1$

$F(2\pi) = 1$

absolute maximum value 3 at $x = \frac{\pi}{2}$
absolute minimum value -1 at $x = \frac{3\pi}{2}$

3. $f(x) = \sqrt[3]{x^2 - 1}$ for $-2 \leqslant x \leqslant 3$

$$f'(x) = \frac{1}{3} \frac{2x}{(x^2-1)^{2/3}}$$

3 critical numbers : $x = 0$, $x = +1$ $x = -1$

$F(-2) = \sqrt[3]{3}$

$F(-1) = 0$

$F(0) = -1$

$F(1) = 0$

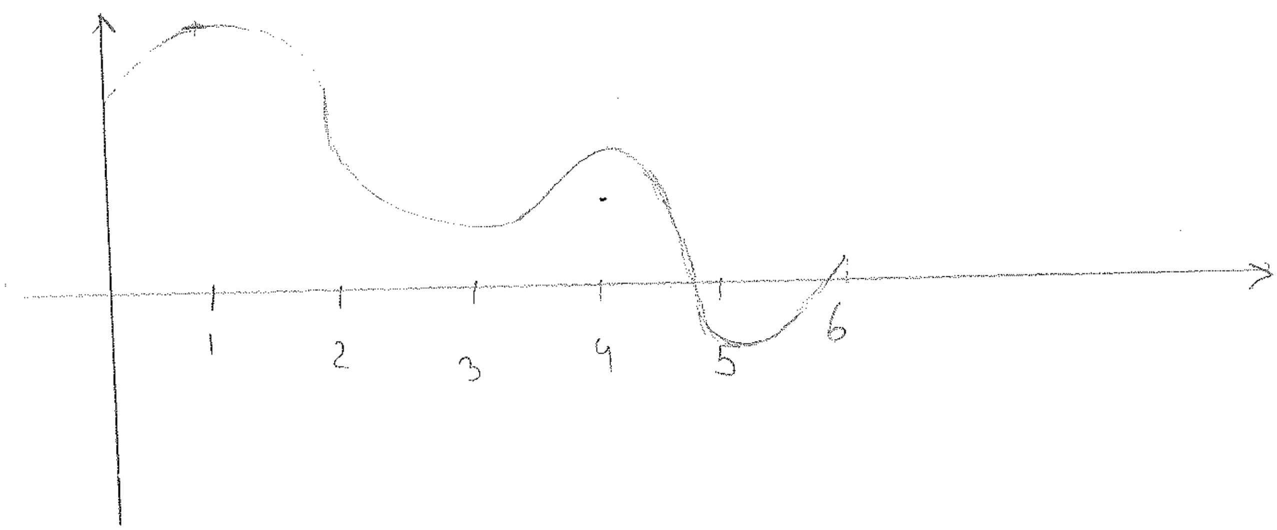
$F(3) = 2$

absolute maximum value : 2 at $x = 3$

absolute minimum value $= -1$ at $x = 0$

Exercise 7. Sketch the graph of a continuous function on $[0,5]$ that satisfies

1. f has a maximum at $x = 1$
2. f has a critical number at $x = 2$ but no local ~~maximum~~ extrema.
3. f has a local minimum at $x = 3$,
4. f has a local maximum at $x = 4$ and the derivative of f does not exist.
5. f has a minimum at $x = 5$



Exercise 8. Find the intervals where the following functions are concave up and concave **down** and identify all inflection points.

1. $f(x) = x^4 - 6x^2$

$$f'(x) = 4x^3 - 12x$$

$$f''(x) = 12x^2 - 12 = 12(x^2 - 1) = 12(x-1)(x+1)$$

	$(-\infty, -1)$	$(-1, 1)$	$(1, \infty)$
$(x-1)$	$-$	$-$	$+$
$(x+1)$	$-$	$+$	$+$
$f''(x)$	$+$	$-$	$+$
	concave upward	concave downward	concave upward.

2 inflection points at $x = -1$ and $x = +1$

2. $f(x) = (x^2 - 1)^5$

$$f'(x) = 5(2x)(x^2-1)^4 = 10x(x^2-1)^4$$

$$f''(x) = 10(x^2-1)^4 + 40(2x^2)(x^2-1)^3$$

$$= 10(x^2-1)^3(9x^2-1)$$

$f''(x) = 0$ For $x = +1$, $x = -1$, $x = \frac{1}{3}$, $x = -\frac{1}{3}$

	$(-\infty, -1)$	$(-1, -\frac{1}{3})$	$(-\frac{1}{3}, \frac{1}{3})$	$(\frac{1}{3}, 1)$	$(1, \infty)$
$(x-1)$	$-$	$-$	$-$	$-$	$+$
$(x+1)$	$-$	$+$	$+$	$+$	$+$
$(3x-1)$	$-$	$-$	$-$	$+$	$+$
$(3x+1)$	$-$	$-$	$+$	$+$	$+$
$f''(x)$	$+$	$-$	$+$	$-$	$+$
	concave upward	concave downward	concave upward	concave downward	concave upward.

$x = 1$, $x = -1$, $x = \frac{1}{3}$, $x = -\frac{1}{3}$ are the 4 inflection points.

3. $f(x) = x^{(2/3)}\sqrt[3]{x-1}$

$$f'(x) = \frac{2}{3}\, x^{(-1/3)} (x-1)^{(1/3)} + \frac{1}{3}\, x^{2/3} (x-1)^{-2/3}$$

$$f''(x) = \frac{-2}{9}\, x^{-4/3}(x-1)^{1/3} + \frac{2}{9}\, x^{-1/3}(x-1)^{-2/3} + \frac{2}{9}\, x^{-1/3}(x-1)^{-2/3} - \frac{2}{9}\, x^{2/3}(x-1)^{-5/3}$$

$$= \frac{x^{-4/3}(x-1)^{-5/3}}{9}\left[-2(x-1)^2 + 2\, x(x-1) + 2x(x-1) - 2x^2\right]$$

$$= \frac{2}{9} x^{-4/3}(x-1)^{-5/3}\left[-(x^2-2x+1) + x^2 - x + x^2 - x - \cancel{2}\, x^2\right]$$

$$= \frac{2}{9} x^{-4/3}(x-1)^{-5/3}\left[\cancel{x} - 1\right]$$

$$= -\frac{2}{9}\, x^{-4/3}(x-1)^{-5/3}(\cancel{x} + 1)$$

$f''(x) = 0$ for $x = 0$, $x = 1$

	$(-\infty, 0)$	$(0, 1)$	$(1, \infty)$
$x^{-4/3}$	+	+	+
$(x-1)^{-5/3}$	−	−	+
$-\frac{2}{9}$	−	−	−
$f''(x)$	+	+	−

concave up   concave down.

$x = 1$: inflection point.

Exercise 9. Let $f(x) = x^5 - 10x^3 + 1$

1. Determine over which intervasl is f increasing? decreasing?

$$f'(x) = 5x^4 - 30x^2$$
$$= 5x^2(x^2 - 6)$$

critical points : $x = 0$, $x = \sqrt{6}$, $x = -\sqrt{6}$

	$(-\infty, -\sqrt{6})$	$(-\sqrt{6}\ \ 0)$	$(0, \sqrt{6})$	$(\sqrt{6}, \infty)$
x^2	+	+	+	+
$x-\sqrt{6}$	−	−	−	+
$x+\sqrt{6}$	−	+	+	+
$f'(x)$	+	−	−	+
f	$\nearrow$	$\searrow$	$\searrow$	$\nearrow$

f increasing on $(-\infty, -\sqrt{6})$ and on $(\sqrt{6}, \infty)$ f decreasing on $(-\sqrt{6}, \sqrt{6})$

2. What are the critical numbers of f? Does the function f has local maxima, local minima at the critical numbers.

critical numbers are $x=0$, $x = \sqrt{6}$, $x = -\sqrt{6}$

local maxima $= f(-\sqrt{6}) = 1 + 24\sqrt{6}$ at $x = -\sqrt{6}$

local minima $= f(\sqrt{6}) = -24\sqrt{6} + 1$ at $x = \sqrt{6}$

3. Determine over which interval is f concave upward? concave downward? Find the inflection points.

$$f''(x) = 20x^3 - 60x$$
$$= 20(x)(x^2 - 3)$$

$f''(x) = 0$ at $x = 0$, $x = \sqrt{3}$, $x = -\sqrt{3}$

	$(-\infty, -\sqrt{3})$	$(-\sqrt{3}, 0)$	$(0, \sqrt{3})$	$(\sqrt{3}, \infty)$
$20x$	$-$	$-$	$+$	$+$
$x - \sqrt{3}$	$-$	$-$	$-$	$+$
$x + \sqrt{3}$	$-$	$+$	$+$	$+$
$f''(x)$	$-$	$+$	$-$	$+$
	concave downward	concave upward	concave downward	concave downward

inflechon points at $x = 0$, $x = \sqrt{3}$, $x = -\sqrt{3}$

Exercise 10. Let f be a 2 times differentiable function such that

x	$f(x)$	$f'(x)$	$f''(x)$
1	3	$f'(x) = 0$	$f''(x) = 4$
4	2	$f'(x) = 0$	$f''(x) = -2$
2	-1	$f'(x) = 0$	$f''(x) = 0$
3	0	$f'(x) DNE$	$f''(x) DNE$

At which critical points of f is there a local extrema? Specify whether they are local maxima or local minima. Illustrate each case by a figure.

Local minimum value : 3 at $x = 1$

local maximum value 2 at $x = 4$

Not enough information to conclude for $x = 2$, and $x = 3$