Section 2.2

1 Differential equations

Definition (1.1): A **differential equation** is an equation involving derivatives.
An equation that describes some physical phenomena is often called a **mathematical model**.

Examples:

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Definitions (1.3):

- A differential equation involving only ordinary differential is called **ordinary differential equation**, (ODE). A differential equation involving partial derivatives is a **partial differential equation** (PDE).
- The order of the highest order derivatives present in the equation is the **order** of the equation.

Exercise 1. Classify the following differential equations as ordinary or partial differential equation, give the order.

1. $y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}^4y}{\mathrm{d}x^4}\right) = 1 + y^2.$
2. $\frac{\partial^2 y}{\partial x \partial y} = y$
3. $x^2\frac{\mathrm{d}y}{\mathrm{d}x} + \left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)^5 + \frac{\mathrm{d}^3y}{\mathrm{d}x^3} = \sin x.$
4. $\frac{\mathrm{d}^2y}{dx^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} - 10y = e^{2x} + \sin x + x^2$
5. $\frac{\partial^2 y}{\partial x^2} = \frac{\partial y}{\partial t}.$

2 Solutions

Definition: A function f is a **solution** to a differential equation on an interval I if f satisfies the differential equation for all x in I.

Exercise 2. Given the differential equation

$$2t^2y'' + 3ty' - y = 0, \qquad t > 0$$

1. Show that $y = A\sqrt{t} + Bt^{-1}$ is solution to the differential equation for any real A and B.

2. Find the solution to the differential equation that satisfies the initial conditions

$$y(1) = 0, \quad y'(1) = 2.$$

Definition (1.2): An expression that contains all the possible solutions of the differential equation is called **general solution**.
The graphs of the solutions are called **integral curves**.

Definition (1.2): Given a differential equation of order n

$$y^{(n)} = f(x, y, y', y'', \cdots, y^{(n-1)})$$

by an **initial value problem**, we mean: Find a solution to the differential equation that satisfies the **initial conditions**

$$y(x_0) = y_0, \quad \frac{\mathrm{d}y}{\mathrm{d}x}(x_0) = y_1, \quad \cdots, \quad \frac{\mathrm{d}y^{(n-1)}}{\mathrm{d}x^{(n-1)}} = y_{n-1}$$

3 Separable equations

Definiton (2.2): A **separable equation** is a first order differential equation that can be written in the form

$$M(x)\mathrm{d}x + N(y)\mathrm{d}y$$

Exercise 3. Determine whether the differential equations are separable. Find the general solution to the separable equations. If possible, find explicit solutions.

1. $\dfrac{dy}{\mathrm{d}x} = \dfrac{x^3}{y}$.

2. $\dfrac{\mathrm{d}y}{\mathrm{d}x} = x\,\mathrm{e}^{y+x^2}$

3. $(1+x^2)\dfrac{\mathrm{d}y}{\mathrm{d}x} + x\sqrt{y} = 0$

4. $\dfrac{\mathrm{d}y}{\mathrm{d}x} = x + y$

Method for solving separable equations:

- Write the equation in the form

$$M(x)\mathrm{d}x = -N(y)\mathrm{d}y$$

- Integrate both sides

$$\int N(y)\mathrm{d}y = -\int M(x)\mathrm{d}x$$

This gives an implicit solution to the differential equation.

- If possible, solve the implicit relation for y to find an explicit solution.

Exercise 4. Solve the initial value problems. If possible, find explicit solutions. Determine at least approximately the interval in which the solution is defined.

1. $e^y y' = 1 + e^y, \quad y(1) = 0.$
2. $y' = \dfrac{2x}{y + x^2 y} \quad y(0) = -2.$