University of Washington

Sections 3.5 – Method of undetermined coefficients

1 Particular solutions

Exercise 1. Find the solutions to the differential equation

$$y'' - 5y' + 6y = 1$$

Theorem 3.5.2: Given a **linear non homogeneous** differential equation

$$y'' + p(x)y' + q(x)y = g(x)$$

Let y_p be a particular solution to the non homogeneous equation on an interval (a, b).
Let $\{y_1, y_2\}$ a fundamental set of solutions on (a, b) of the **corresponding homogeneous** differential equation

$$y'' + p(x)y' + q(x)y = 0$$

Then the general solution to the **non homogeneous** equation is

$$y = y_p + c_1 y_1 + c_2 y_2$$

for any constant c_1 and c_2.

Procedure for solving non homogeneous linear second order differential equations:

- Find the general solution y_h to the **corresponding homogeneous equation**.
- Find a particular solution y_p to the **non homogeneous equation**.
- The general solution to the **non homogeneous equation** is

$$y = y_p + y_h.$$

2 Method of undetermined coefficients (3.5)

For linear differential equation **with constant coefficients, non homogeneous**.

Exercise 2. Find the general solution to

$$y'' - 5y' + 6y = e^{4x}.$$

Exercise 3. Find the general solution to

$$y'' - 5y' + 6y = e^{2x}$$

Exercise 4. Find the general solution to

$$y'' - 4y' + 4y = \mathrm{e}^{2x}$$

> **Theorem:** Given a linear second order differential operator with constant coefficients.
>
> $$ay'' + by' + cy = \mathrm{e}^{\alpha x}$$
>
> A particular solution to the differential equation is in the form $y = kx^s\,\mathrm{e}^{ax}$.
> $s = 0$ if $\mathrm{e}^{\alpha x}$ is not a solution to the corresponding homogeneous problem.
> $s = 1$ if $\mathrm{e}^{\alpha x}$ is a solution to the corresponding homogeneous problem.
> $s = 2$ if $\mathrm{e}^{\alpha x}$ and $x\,\mathrm{e}^{ax}$ are solutions to the corresponding homogeneous problem.

Exercise 5.

- Find a particular solution to $y'' - 5y' + 6y = x^2 + 3$.

> **Theorem:** If $ay'' + by' + cy = P_n(x)$ is a linear second order differential operator with constant coefficients and P_n is a polynomial function of degree n.
> A particular solution is in the form $y = x^s p_n(x)$ where
>
> - p_n is a polynomial function of degree n
> - $s = 0$ if the constant functions are solutions to the corresponding homogeneous differential equation.
> - $s = 1$ if the constant functions are solutions to the corresponding homogeneous differential equation,
> - $s = 2$ if the constant functions and the function x is solution to the homogeneous equation

- Find a particular solution to $y'' - 5y' + 6y = 20\sin(2x)$.

> **Theorem:** If $ay'' + by' + c = a\cos(\alpha x) + b\sin(\alpha x)$ is a linear second order differential operator with constant coefficients.
> A particular solution is in the form $y = x^s(A\cos(\alpha x) + B\sin(\alpha x))$.
> $s = 1$ is $\cos\alpha x$ is solution to the homogeneous solution, $s = 0$ otherwise.

Remark:

Procedure for finding a particular solution: Let

$$y'' + py' + qy = g(x)$$

be a second order differential equation **with constant coefficients**.

- Solve the **corresponding homogeneous problem**, y_h.
- On the basis of g, guess the appropriate form of the particular solution y_p of the **non homogeneous equation**.
- Use the differential equation to find relations between coefficients appearing in the particular solution y_p. (You need to look at y_h to determine the value of s.)
- Solve the system and find the coefficients.

$g(x)$	Guess
e^{ax}	$x^s(Ae^{ax})$
$\cos ax$	$x^s(A\cos ax + B\sin ax)$
$\sin ax$	$x^s(A\cos ax + B\sin ax)$
$P_n(x) = a_nx^n + \cdots + a_1x + a_0$	$x^s(A_nx^n + \cdots + A_1x + A_0)$
$e^{ax}\cos bx$ or $e^{ax}\sin bx$	$x^s(Ae^{ax}\cos bx + Be^{ax}\sin bx)$
$P_n(x)e^{ax}$	$x^s(A_nx^n + \cdots + A_1x + A_0)e^{ax}$
$P_n(x)\cos ax$ or $P_n(x)\sin ax$	$x^s((A_nx^n + \cdots + A_1x + A_0)\cos ax$ $+(B_nx^n + \cdots + B_1x + B_0)\sin ax)$
$P_n(x)e^{ax}\cos bx$ or $P_ne^{ax}\sin bx$	$x^s((A_nx^n + \cdots + A_1x + A_0)e^{ax}\cos bx$ $+(B_nx^n + \cdots + B_1x + B_0)e^{ax}\sin bx)$

Exercise 6. Find a valid guess for a particular solution to the differential equation
$y'' - 5y' + 6y = e^{3x} + x^2e^{2x} + \sin 2x + e^{3x}\cos 2x$
Do not solve for the coefficients.