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Section 6-1, Definition of Laplace transform

Definition: Let f be a function on $[0, \infty)$. The Laplace transform of f is the function $\mathcal{L}\{f\}$ defined by the integral

$$\mathcal{L}\{f\}(s) = \int_0^\infty \mathrm{e}^{-st} f(t)\mathrm{d}t = \lim_{N\to\infty} \int_0^N \mathrm{e}^{-st} f(t)\mathrm{d}t$$

The domain of $\mathcal{L}\{f\}$ is all the values of s for which the integral exists.

Exercise 1. Find the Laplace transforms of the following functions and give their domains.

1. $f(t) = 1$.
2. $f(t) = t$.
3. $f(t) = \mathrm{e}^{3t}$.
4. $f(t) = \sin t$.
5. $f(t) = e^{2t} \cos t$.

Definition (p 310): A function is piecewise continuous on an interval $a \leqslant t \leqslant b$ if the interval $[a, b]$ can be partitioned by a finite number of points $a = t_0 < t_1 < \cdots < t_n = b$ so that

1. f is continuous on each open interval (t_i, t_{i+1}).
2. f approached a finite limit as the end points of each interval are approached from within the subinterval.

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Theorem (6.1.2): Suppose that

1. f is piecewise defined function on the interval $[0, A]$ for any positive A.
2. $|f(t)| \leqslant Ke^{at}$ for $t \geqslant M$. In this inequality, K, a, and M are real constants, K and M are necessarily positive.

Then the Laplace transform

$$\mathcal{L}\{f\}(s) = \int_0^\infty \mathrm{e}^{-st} f(t)\mathrm{d}t = \lim_{N\to\infty} \int_0^N \mathrm{e}^{-st} f(t)\mathrm{d}t$$

exists for $s > a$.

Exercise 2. Find the Laplace transform of $f(t) = \begin{cases} 1 & 0 \leqslant t < 1, \quad \text{and} \quad t > 10 \\ 3 & 1 \leqslant t \leqslant 10. \end{cases}$

Theorem ((6) p 314): Let f_1 and f_2 two functions such that their Laplace transform exist for $s > a_1$ and $s > a_2$ respectively. Then for $s > Max(a_1, a_2)$, for any real constant c_1 and c_2,

$$\mathcal{L}\{c_1 f_1 + c_2 f_2\} = c_1 \mathcal{L}\{f_1\} + c_2 \mathcal{L}\{f_2\}$$

Exercise 3. Find the Laplace transform of $f(t) = e^{5t} + 3e^{2t}\cos t$.