Section 6-2

Theorem: Let f be a continuous function on $[0, \infty)$ and $f'(t)$ be a piecewise continuous function on $[0, \infty)$ with both of exponential order α. Then for $s > \alpha$,

$$\mathcal{L}\left\{f'\right\}(s) = s\mathcal{L}\left\{f\right\}(s) - f(0).$$

Exercise 1. Prove the Theorem.

Exercise 2. Find the Laplace transform of e^{ax} where a is a real number.

Theorem: Let $f(t),\ f'(t), \cdots, f^{(n-1)}(t)$ be continuous on $[0, \infty)$ and let $f^{(n)}(t)$ be a piecewise continuous function on $[0, \infty)$, with all these functions of exponential order α. Then for $s > \alpha$,

$$\mathcal{L}\left\{f^{(n)}\right\}(s) = s^n\mathcal{L}\left\{f\right\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0).$$

Exercise 3. Write an expression for the Laplace transform of the second derivative.

Exercise 4. Find the Laplace transform of $\sin 2x$.

Exercise 5. Let y be a solution to the initial value problem

$$y'' - 4y = 1 \qquad y(0) = 0, \quad y'(0) = 1.$$

Find the Laplace transform of y.

> **Definition:** Given a function $F(s)$, if there is a function $f(t)$ that is continuous on $[0, \infty)$ and satisfies
>
> $$\mathcal{L}\{f\}(s) = F(s)$$
>
> then $f(t)$ is the inverse Laplace transform of $F(s)$, written $f = \mathcal{L}^{-1}\{F\}$.

Solve the previous initial value problem.

Exercise 6. Find the inverse Laplace transform of $F(s) = \dfrac{3s^2 + s}{(s+1)(s-1)^2}$.

Laplace Transform Formula:

$f(t)$	$\mathcal{L}\{f\}(s)$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
$t^n \quad n = 1, 2, \cdots$	$\frac{n!}{s^{n+1}}$
$\sin bt$	$\frac{b}{s^2+b^2}$
$\cos bt$	$\frac{s}{s^2+b^2}$
$e^{at}\, t^n \quad n = 1, 2, \cdots$	$\frac{n!}{(s-a)^{n+1}}$
$e^{at} \sin bt$	$\frac{b}{(s-a)^2+b^2}$
$e^{at} \cos bt$	$\frac{s-a}{(s-a)^2+b^2}$

$$\mathcal{L}\left\{e^{at} f(t)\right\} = \mathcal{L}\{f(t)\}(s-a)$$

$$\mathcal{L}\left\{f'\right\}(s) = s\mathcal{L}\{f\}(s) - f(0).$$

$$\mathcal{L}\left\{f''\right\}(s) = s^2\mathcal{L}\{f\}(s) - f'(0) - sf(0).$$

$$\mathcal{L}\left\{f^{(n)}\right\}(s) = s^n\mathcal{L}\{f\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0).$$

$$\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \frac{d^n F}{ds^n}(s).$$

Exercise 7. Solve the initial value problem

$$y'' + 6y' + 5y = 12\, e^t, \quad y(0) = -1, \quad y'(0) = 7$$

using Laplace transforms.

Method of Laplace transforms: To solve an initial value problem:

1. Take the Laplace transform of both sides of the equation.
2. Use the properties of the Laplace transform and the initial conditions to obtain an equation involving the Laplace transform of the solution.
3. Determine the Laplace transform of the solution $\mathcal{L}\{y\}$ by solving the previous equation for $\mathcal{L}\{y\}(s)$.
4. Determine the solution by taking the inverse Laplace transform of $\mathcal{L}\{y\}$.

Exercise 8. Solve the initial value problem

$$w'' + 4w = 2t, \qquad w(0) = 1, \quad w'(0) = -1$$

Exercise 9. Find a solution to the initial value problem

$$y'' + 3ty' - 6y = 1, \qquad y(0) = 0, \quad y'(0) = 0$$

Exercise 10. (24p325) Solve the initial value problem

$$y'' + 4y = \begin{cases} 1, & 0 \leqslant t < \pi \\ 0 & \pi \leqslant t < \infty \end{cases} \qquad y(0) = 1, \quad y'(0) = 0$$