University of Washington

Sections 6-3, 6.4, 6.5

1 Section 6.3

Definition: The unit step function or Heaviside function is the function u_c defined by

$$u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geqslant c \end{cases} \qquad c \geqslant 0.$$

Exercise 1. Evaluate

1. $u_3(5) =$

2. $u_7(2) =$

Exercise 2. Sketch the graph of the functions

1. $f(t) = u_0(t)$.

2. $g(t) = u_a(t) - u_b(t)$ where a and b are 2 real numbers, $(a < b)$

3. $h(t) = t + (t-1)u_2(t) + 2(t-3)u_3(t)$

Exercise 3. Express the following functions in terms of step functions:

1. $f(t) = \begin{cases} t & 0 \leqslant t < 3 \\ 2t-1 & 3 \leqslant t < 5 \\ t^2 & 5 \leqslant t \end{cases}$

2. $f(t) = \begin{cases} 1 & 0 \leqslant t < 3 \\ e^t & 3 \leqslant t < 4 \\ 0 & 4 \leqslant t \end{cases}$.

> **Theorem 6.3.1:** If $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a \geqslant 0$, and if c is a **positive** constant, then
>
> $$\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}\,\mathcal{L}\{f(t)\} = e^{-cs}\,F(s), \quad s > a.$$
>
> $$\mathcal{L}\{u_c(t)h(t)\} = e^{-cs}\,\mathcal{L}\{h(t+c)\}, \quad s > a.$$
>
> Conversely,
>
> $$\mathcal{L}^{-1}\left\{e^{-cs}\,\mathcal{L}\{f(t)\}\right\} = u_c(t)f(t-c).$$

Exercise 4. Find the Laplace transform of the function

$$f(t) = t + (t-1)u_1(t) + 2tu_3(t)$$

Exercise 5. Let

$$f(t) = \begin{cases} 2t, & 0 \leqslant t < 2 \\ t^2 & 2 \leqslant t < 5 \\ 0 & 5 \leqslant t \end{cases}$$

Find the Laplace transform of f.

Exercise 6. Find the inverse Laplace transform of the functions

$$F(s) = \frac{\mathrm{e}^{-2s}}{s^2 - 2s - 3}, \quad G(s) = \frac{\mathrm{e}^{-5s} + 3}{s^2 + 4s + 5}, \quad H(s) = \frac{2}{(s-3)(s+1)} + \frac{e^{-s}}{s^3}.$$

Exercise 7. Let

$$f(x) = \begin{cases} 1, & 0 \leqslant t < 1 \\ 0, & 1 \leqslant t < 2 \end{cases} \qquad f(t) = f(t+2)$$

Find the Laplace transform of f.

Theorem 6.3.2: If $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a \geqslant 0$, and if c is a constant, then

$$\mathcal{L}\left\{e^{ct} f(t)\right\} = F(s-c), \quad s > a + c.$$

Conversely, if $f(t) = \mathcal{L}^{-1}\{F(s)\}$, then

$$\mathrm{e}^{ct} f(t) = \mathcal{L}^{-1}\{F(s-c)\}.$$

2 Section 6.4

Exercise 8. Find the solutions of the initial value problems

1. $y'' + y = f(t), \qquad y(0) = 0, \quad y'(0) = 1, \quad f(t) = \begin{cases} 1, & 0 \leqslant t < 3\pi \\ 0, & 3\pi \leqslant t \end{cases}$

2. $y'' + 3y' + 2y = u_2(t), \qquad y(0) = 0, \quad y'(0) = 1.$

3 Section 6.5

Definition: The Dirac delta function $\delta(t)$ is characterized by the following properties:

1. $\delta(t) = 0$ if $t \neq 0$. $\delta(0)$ is infinite.
2. $\displaystyle\int_{-\infty}^{\infty} f(t)\delta(t)\mathrm{d}t = f(0)$ for any continuous functions on an interval containing $t = 0$.

Exercise 9. Find the solution of the given initial value problem

1. $y'' + 9y = 3\delta(t - \pi), \qquad y(0) = 1, \quad y'(0) = 0.$

2. $y'' + 2y' + 2y = \delta(t - \pi), \qquad y(0) = 1, \quad y'(0) = 0.$

3. $y'' + 2y' + 3y = \sin t + \delta(t - 3\pi), \qquad y(0) = 0, \quad y'(0) = 0.$

4. $y'' - y' - 2y = 3\delta(t - 1) + e^t, \qquad y(0) = 0, \quad y'(0) = 1.$