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Review for Exam 1, Sections 2.1, 2.2, 2.3, 2.4, 2.5, 2.7, 3.1, 3.2, 3.3, 3.4, 3.5

Exercise 1. (close to 21p8) A pond initially contains 1,000,000 gal of water and an unknown amount of undesirable chemical. Water containing **0.001g** of this chemical per gallon flows into the pond at a rate of 300gal/h. The mixture flows out at the same rate, so the amount of water in the pond remains constant. Assume that the chemical is uniformly distributed throughout the pond.

1. Write a differential equation for the amount of chemical in the pond at any time.

2. How much of the chemical will be in the pond after a long time. Does this limiting amount depend on the amount that was present initially?

Exercise 2. (19p18-19) Your swimming pool containing 60,000 gal of water has been contaminated by 5 kg of a non toxic dye that leaves a swimmer's skin an unattractive green. The pool's filtering system can take water from the pool, remove the dye, and return the water to the pool at a flow rate of 200 gal/min.

1. Write down the initial value problem for the filtering process; let $q(t)$ be the amount of dye in the pool at any time t.

2. Solve the problem.

3. You have invited several dozen friends to a pool party that is scheduled to begin in 4 hours. You have also determined that the effect of the dye us imperceptible if its concentration is less than 0.02 g/gal. Is your filtering system capable of reducing the dye concentration to this level within 4 hours?
4. Find the time T at which the concentration of dye first reaches the value 0.02 g/gal.
5. Find the flow rate that is sufficient to achieve the concentration 0.02g/gal within 4 hours.

Exercise 3. (p24) Determine the order of the equations and state whether they are linear, non linear...

1. $t^2y''(t) + ty'(t) + 2y = \sin t$
2. $(1 + y^2)y'' + ty' + y = \mathrm{e}^t$
3. $\dfrac{\mathrm{d}^4y}{\mathrm{d}t^4} + \dfrac{\mathrm{d}^3y}{\mathrm{d}t^3} + \dfrac{\mathrm{d}^2y}{\mathrm{d}t^2} + \dfrac{\mathrm{d}y}{\mathrm{d}t} + y = 1.$
4. $\dfrac{\mathrm{d}y}{\mathrm{d}t} + ty^2 = 0.$
5. $\dfrac{\mathrm{d}^2y}{\mathrm{d}t^2} + \sin(t + y) = \sin t.$

Exercise 4. (p24-25) Verify that the given functions are solutions to the differential equations:

1. $y'' - y = 0 \quad y_1(x) = \mathrm{e}^x, \quad y_2(x) = \frac{e^x - e^{-x}}{2}.$
2. $2t^2y'' + 3ty' - y = 0 \quad t > 0, \quad y_1(t) = \sqrt{t}, \quad y_2(t) = t^{-1}.$
3. $t^2y'' + 5ty' + 4y = 0, \quad t > 0, y_1(t) = t^{-2} \quad y_2(t) = t^{-2}\ln(t).$

Exercise 5. Find the general solution to (Find an explicit solution if possible)

1. (2p40) $y' - 2y = t^2\,\mathrm{e}^{2t}$
2. (1p40) $y' + 3y = t + \mathrm{e}^{-2t}.$
3. (4p40) $y' + \dfrac{1}{t}y = 3\cos(2t) \quad t > 0.$
4. (6p40) $ty' + 2y = \sin t, \quad t > 0.$
5. (8p40) $(1 + t^2)y' + 4ty = (1 + t^2)^{-2}.$
6. (7p48) $y' = \dfrac{x - \mathrm{e}^{-x}}{y + \mathrm{e}^y}$

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7. (8p48) $y' = \dfrac{x^2}{1+y^2}$

Exercise 6. Solve the initial value problem

1. (13p40) $y' - y = 2t\,\mathrm{e}^{2t}, \quad y(0) = 1$
2. (15p40) $ty' + 2y = t^2 - t + 1, \quad y(1) = 0.5, \quad t > 0$
3. (18p40) $ty' + 2y = \sin t, \quad y(\pi/2) = 1, \quad t > 0$
4. (20p40) $ty' + (1+t)y = t, \quad y(\ln(2)) = 1, \quad t > 0$
5. (9p48) $y = (1-2x)y^2, \quad y(0) = \dfrac{-1}{6}.$
6. (11p48) $x + y\,\mathrm{e}^{-x}\,y' = 0, \quad y(0) = 1.$
7. (13p48) $y' = \dfrac{2x}{y + x^2 y}, \quad y(0) = -2.$

Exercise 7. (23p48) Solve the initial value problem and determine where the solution attains its minimum value.

$$y' = 2y^2 + xy^2, \quad y(0) = 1.$$

Exercise 8. Find the value of y_0 for which the solution to the initial value problem

$$y' = y = 1 + 3\sin t, \qquad y(0) = y_0$$

remains finite as $t \to 0$.

Exercise 9. (4p60) A tank with a capacity of 500 gal originally contains 200 gal of water with 100lb of salt in solution. Water containing 1lb of salt per gallon is entering at a rate of 3 gal/min, and the mixture is allowed to flow out of the tank at a rate of 2 gal/min. Find the amount of salt in the tank at any time prior to the instant when the solution begins to overflow. Find the concentration (in pound per gallon) of salt in the tank when it is on the point of overflowing. Compare the concentration with the theoretical limiting concentration if the tank had infinite capacity.

Exercise 10. (3p60) A tank originally contains 100 gal of fresh water. Then water containing 0.5 lb of salt per gallon is poured into the tank at the rate of 2 gal/min. and the mixture is allowed to leave at the same rate.

After 10 minutes, the process is stopped, and fresh water is poured into the tank at a rate of 2 gal/min with the mixture again leaving at the same rate. Find the amount of salt in the tank at the end of an additional 10 min.

Exercise 11. (16p62) Newton's law of cooling states that the temperature of an object changes at a rate proportional to the difference between its temperature and that of its surroundings. Suppose that the temperature of a cup of coffee obeys the Newton's law of cooling. If the coffee

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has a temperature of 200°F when freshly poured, and 1 minutes later has cooled to 190°F in a room at 70°F, determine when the coffee reaches a temperature of 150°F.

Exercise 12. (20, 21 p64-65) A ball with mass 0.15kg is thrown upward with initial velocity 20 m/s from the roof of a building 30m high. Neglect air resistance.

1. Find the maximum height above the ground that the ball reaches.
2. Assuming that the ball misses the building on the way down, find the time that it hits the ground.

Assume that the conditions are as previously except that there is a force due to air resistance of magnitude $|v|/30$ directed opposite to the velocity, where the velocity v is measured in m/s.

1. Find the maximum height above the ground that the ball reaches.
2. Assuming that the ball misses the building on the way down, find the time that it hits the ground.

Exercise 13. (1,2,3, 6 p76) Determine an interval in which the solution of the given initial value problem is certain to exist:

1. $(t-3)y' + \ln(t)y = 2t, \quad y(1) = 2$
2. $t(t-4)y' + y = 0, \quad y(2) = 1$
3. $y' + (\tan t)y = \sin t, \qquad y(\pi) = 0$
4. $(\ln t)y' + y = \cot t, \qquad y(2) = 3.$.

Exercise 14. (7, 10 p76) State where in the ty-plane the hypotheses of the theorem 2.4.2 are satisfied:

$$y' = \frac{t-y}{2t+5y}$$

$$y' = (t^2+y^2)^{(3/2)}$$

Exercise 15. (14p76) Solve the initial value problem and determine how the interval in which the solution exists depends on the initial value y_0:

$$y' = 2ty^2, \quad y(0) = y_0.$$

Exercise 16. (4,10,12 p 88-89) Sketch the graph of $f(y)$ versus y, determine the equilibrium solutions, and classify each one as asymptotically stable, unstable or semi stable. Draw the phase line, and sketch several graphs of solutions in the ty-plane:

1. $y'(t) = e^y - 1, \quad -\infty < y_0 < \infty.$
2. $y'(t) = y(1-y^2), \quad -\infty < y_0 < \infty.$

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3. $y' = y^2(4 - y^2)$.

Exercise 17.

1. Find approximate values of the solution of the given initial value problem at t =0.1, 0.2, 0.3, and 0.4 using Euler method with $h = 0.1$.
2. Repeat part (a) with $h = 0.05$. Compare the result with those found in (a).
3. Find the solution $\phi(t)$ of the given initial value problem and evaluate $\phi(t)$ at t =0.1, 0.2, 0.3, and 0.4. Compare these values with the results of (a), (b).

   (a) (1p110) $y' = 3 + t - y, \qquad y(0) = 1,$

   (b) (2p110) $y' = 2y - 1, \qquad y(0) = 1.$

Exercise 18. Find the longest interval in which the solution of the initial value problems is certain to exist and be unique.

1. (9p155) $(t^2 - 4t)y'' + 3ty' + 4y = 2, \qquad y(3) = 0, \quad y'(3) = -1.$
2. (11p156)$(x - 3)y'' + xy' + \ln(|x|)y = 0, \qquad y(1) = 0, \quad y'(1) = 1.$
3. (12p156) $(x - 2)y'' + y' + (x - 2)(\tan x)y = 0, \qquad y(3) = 1, \quad y'(3) = 2.$

Exercise 19. (13p156) Verify that $y_1(t) = t^2$, and $y_2(t) = t^{-1}$ are two solutions of the differential equation $t^2t'' - 2y = 0$ for $t > 0$. Then show that $y = c_1y_1 + c_2y_2$ is also solution of this equation for any constant c_1 and c_2.

Exercise 20. Verify that $y_1(t) = 1$ and $y_2(t) = t^{1/2}$ are solutions to the differential equation $yy'' + (y')^2 = 0$ for $t > 0$. Then show that $y = c_1 + c_2t^{1/2}$ is not, in general, a solution of this equation. explain why this result does not contradict Theorem 3.2.2 (Theorem of superposition).

Exercise 21. (18p156) If the Wronskian W of f and g is t^2e^t and if $f(t) = t$, find $g(t)$.

Exercise 22. y_1 and y_2 are solutions to the given differential equation. Do they constitute a fundamental set of solutions?

1. (24p155) $y'' + 4y' = 0, \qquad y_1(x) = \cos 2x, \quad y_2(x) = \sin 2x.$
2. $y'' + 4y = 0, \qquad y_1(x) = 5\cos^2 x - \sin^2 x - 2, \qquad y_2 = \cos^2 x - \sin^2 x$

   Hint: They are not a fundamental set of solutions, $W\{y_1, y_2\}(x) = 0$ *for any* x

Exercise 23. Find a fundamental set of solutions of the given equation:

1. (Use 2p144) $y'' + 3y' + 2y = 0$

2. (Use 11p164) $y'' + 6y' + 13y = 0$
3. (Use 4p172) $4y'' + 12y' + 9y = 0$
4. (Use 5p144) $y'' + 5y' = 0$
5. (Use 7p164) $y'' - 2y' + 2y = 0$

For such a question, you need to find 2 solutions and justify why they are a fundamental set of solutions by evaluating the Wronskian, for example.

Exercise 24. Solve the given initial value problem:

1. (14p173) $y'' + 4y' + 4y = 0, \quad y(-1) = 2, y'(-1) = 1$
2. (18p164) $y'' - 2y' + 5y = 0, \quad y(0) = 1, y'(0) = 0$
3. (10p144) $y'' + 4y' + 3y = 0, \qquad y(0) = 2, \quad y'(0) = -1$
4. (12p173) $y'' - 6y' + 9y = 0, \qquad y(0) = 0, \quad y'(0) = 2.$
5. (20p164) $y'' + y = 0, \qquad y(\pi/3) = 2, \quad y'(\pi/3) = -4$
6. (11p144) $6y'' - 5y' + y = 0, \qquad y(0) = 4, \quad y'(0) = 0$

Exercise 25. Find a second solution of the given differential equation such that $\{y_1, y_2\}$ is a fundamental set of solution on the given interval. (Check that is a fundamental set of solutions)

1. (23p174) $t^2y'' - 4ty' + 6y = 0, \quad t > 0, \quad y_1 = t^2.$
2. (25p174) $t^2y'' + 3ty' + y = 0, \quad t > 0, \quad y_1(t) = t^{-1}.$
3. (26p174) $t^2y'' - t(t+2)y' + (t+2)y = 0, \quad t > 0, \quad y_1 = t.$
4. (28p174) $(x-1)y'' - xy' + y = 0, \qquad x > 1, \quad y_1(x) = e^x.$

Exercise 26. (21p144) Solve the initial value problem $y'' - y' - 2y = 0, \qquad y(0) = \alpha, \quad y'(0) = 2$. Then find α so that the solution approaches zero as $t \to 0$.

Exercise 27. (23p144) Determine the values of α for which all the solutions tend to zero as $t \to \infty$; also determine the values of α, if any, for which all (non zero solutions become unbounded as $t \to \infty$.

$$y'' - (2\alpha - 1)y' + \alpha(\alpha - 1)y = 0$$

Exercise 28. (25p164) Consider the initial value problem

$$y'' + 2y' + 6y = 0, \qquad y(0) = 2, \quad y'(0) = \alpha \geqslant 0$$

1. Find the solution $y(t)$ of this problem.

2. Find α such that $y = 0$ when $t = 1$.

3. Find, as a function of α, the smallest positive value of t for which $y = 0$

4. Determine the limit of the expression found in part3. as $\alpha \to \infty$.

Exercise 29. Find the general solution of the given differential equation or initial value problem.

1. (1p184) $y'' - 2y' - 3y = 3\,\mathrm{e}^{2t}$

2. (2p184) $y'' + 2y' + 5y = 3\sin 2t$.

3. (7p184) $y'' + 9y = t^2 e^{3t} + 6$

4. (9p184) $2y'' + 3y' + y = t^2 + 3\sin t$.

5. (19p184) $y'' + 4y = 3\sin 2t, \quad y(0) = 2, \quad y'(0) = -1$.

6. (15p184) $y'' + y' - 2y = 2t, \quad y(0) = 0, y'(0) = 1$.

7. (18p184) $y'' - 2y' - 3y = 3te^{2t}, \qquad y(0) = 1, y'(0) = 0$.

8. $y'' - 2y' - 3y = 2te^{3t}, \qquad y(0) = 1, y'(0) = 0$.

Exercise 30. Determine a suitable for the particular solution $Y(t)$ if the method of undetermined coefficients is to be used.

1. (21p184) $y'' + 3y' = 2t^4 + t^2 e^{-3t} + \sin 3t$

2. (23p184) $y'' - 5y' + 6y = e^t \cos 2t + e^{2t}(3t + 4)\sin t$

3. (24p184) $y'' + 2y' + 2y = 3e^{-t} + 2e^{-t}\cos t + 4e^{-t}t^2 \sin t$

4. (27p185) $y'' + 3y' + 2y = e^t(t^2 + 1)\sin t + e^{-t}\cos t + 4e^t$

Exercise 31. Use the method of variation of parameters to find a particular solution of

1. (5p190) $y'' + y = \tan x, \quad t \in (0, \pi/2)$ `Hint: An antiderivative of` $\sec t$ `is` $\ln(\sec t + \tan t)$`.`

2. (7p190) $y'' + 4y' + 4y = t^{-2}e^{-2t}$.

3. (1p190) $y'' - 5y' + 6y = 2\,\mathrm{e}^t$.

4. (13p190) $t^2 y'' - 2y = 3t^2 - 1, \qquad t > 0, \quad y_1(t) = t^2, \quad y_2(t) = t^{-1}$.

5. (16p190) $(1 - t)y'' + ty'' - y = 2(t - 1)^2 e^{-t}, \qquad 0 < t < 1, \quad y_1(t) = e^t, \quad y_2(t) = t$.