

Last Name (PRINT): _____
First Name (PRINT): _____

**Autumn 2014 – Introductory Real Analysis
First Examination**

Instructions

1. The use of all electronic devices is prohibited.
2. Present your solutions in the space provided. Show all your work neatly and concisely. Clearly indicate your final answer. You will be graded not merely on the final answer, but also on the quality and correctness of the work leading up to it.

Scholastic dishonesty will not be tolerated.
The work on this test is my own.

Signature: _____

Questions	1	2	3	4	5	Total
Grade:						

Exercise 1.

1. Prove that for any real a , $a * 0 = 0$.

This is proposition 2.2, You may use any result stated before proposition 2.2

2. Prove that for any real a , $(-a) = (-1) * a$.

This is proposition 2.4, You may use any result stated before proposition 2.4

Exercise 2. Prove that for any real a and b , $|a + b| \leq |a| + |b|$.

This result is the triangular inequality. You may use any result mentioned before Proposition 7.

Exercise 3. Let $S = \{x \in \mathbb{R} \mid x^2 < x\}$.

Find the least upper bound and prove that is really is the least upper bound.

Exercise 4. Let a_n be a convergent sequence. Assume that $A = \lim_{n \rightarrow \infty} a_n$ is positive. ($A > 0$)

1. Prove that there exists N_1 such that if $n > N_1$ then $a_n > \frac{A}{2}$.

2. Prove using the definition of the limit and the previous question, that $\lim_{n \rightarrow \infty} \frac{1}{a_n} = \frac{1}{A}$.

This result is a preliminary result to Theorem 29. You may use any result stated prior to Theorem 29

Exercise 5. Let a_n be a convergent sequence.

Prove that the sequence a_n converges to 0 if and only if the sequence $\lim_{n \rightarrow \infty} \frac{1}{a_n} = \infty$.

1 Axioms of \mathbb{R}

Axiom 1. \mathbb{R} is a commutative field.

Definition 1. $F + *$ is a commutative field if

1. For any a, b, c in F , $a + (b + c) = (a + b) + c$, and $a * (b * c) = (a * b) * c$ (associativity).
2. For any a and b in F , $a + b = b + a$ and $a * b = b * a$ (commutativity).
3. For any a, b, c in F , $a * (b + c) = a * b + a * c$ (distributivity).
4. There exists an element written 0 such that for any a in F , $a + 0 = a$ (additive identity)
5. There exists an element written 1 such that for any a in F , $a * 1 = a$ (multiplicative identity)
6. For any a in F , there exists an element written $-a$ such that $a + (-a) = 0$ (additive inverse)
7. For any a in F except 0 , there exists an element written a^{-1} such that $a * (a^{-1}) = 1$ (multiplicative inverse).

Proposition 2. 1. The additive inverse is unique.

2. For any real number a , $0 * a = 0$.
3. $(-1)(-1) = 1$.
4. The additive inverse of $a + b$ is $(-a) + (-b)$.
5. for any real number a , $-a = (-1) * a$.

Axiom 2. \mathbb{R} is totally ordered:
There is a relation $>$ that satisfies

1. If $a > 0$ and $b > 0$, then $a + b > 0$
2. If $a > 0$ and $b > 0$, then $a * b > 0$
3. For each a only one of the following is true
 - (a) $a > 0$,
 - (b) $a = 0$,
 - (c) $-a > 0$

Definition 3. We say that $a < b$ if $b - a > 0$

Proposition 4. Given three real numbers a, b, c ,

1. $0 < a^2$ if $a \neq 0$
2. If $a < b$ and $b < c$ then $a < c$
3. If $a < b$ and $0 < c$, then $ac < bc$
4. If $a < b$, for any c , $a + c < b + c$

Proposition 5. 1. If a is a positive real number, then its multiplicative inverse a^{-1} is positive.
2. For any real numbers such that $0 < a < b$, then $b^{-1} < a^{-1}$.

Definition 6. The absolute value of a , written $|a|$ is

- $|a| = a$ if $a > 0$
- $|a| = -a$ if $a < 0$
- $|a| = 0$ if $a = 0$

Proposition 7. For any a, b in \mathbb{R} ,

- $|a * b| = |a| * |b|$
- $|a + b| \leq |a| + |b|$
- $||a| - |b|| \leq |a - b|$

Proposition 8. For any real numbers such that $a < b < c$, $|b| \leq \max(|a|, |c|)$.

Axiom 3. (Axiom of continuity) Suppose that all real numbers are separated into two collections which we denote by L and R , in such a way that

1. Every number is either in L or in R .
2. Each collection contains at least 1 element.
3. If a is in L and b is in R , then $a < b$.

then there is a number c in \mathbb{R} such that all numbers less than c are in L and all numbers greater than c are in R .

Proposition 9. The cut number c is unique.

Theorem 10. (Archimedean law of real numbers) Let a and b be 2 positive real numbers. There exists a positive integer n such that $b < na$.

Proposition 11. If a real number y satisfies that $0 \leq y \leq \frac{1}{n}$ for any natural number n , then $y = 0$.

Proposition 12. For any positive real number y , there exists a real number c such that $c^2 = y$.

Theorem 13. Any system that satisfies Axiom 1, 2, and 3 is isomorphic to \mathbb{R}

2 Axioms of \mathbb{N} (2.3)

Axiom 4. The natural numbers are the smallest class of real numbers that satisfies

1. 1 is an element of \mathbb{N}
2. If n is an element of \mathbb{N} , $n + 1$ is an element of \mathbb{N}

Proposition 14. Any natural number either equals to 1 or is greater than 1.

Proposition 15. For any 2 natural numbers $p < n$, there exist a natural number m such that $n = p + m$.

Proposition 16. Let n be a natural number. there is no natural number between n and $n + 1$.

Proposition 17. If S is a class of positive integers containing at least 1 element, it contains a smallest element.

3 Rational & irrational numbers (2.5)

Definition 18. A rational number is a number that can be written in the form $\pm \frac{p}{q}$, where p and q are 2 natural numbers.
An irrational number is a number that is not rational

Theorem 19. The number $\sqrt{2}$ is irrational.

Theorem 20. Given to numbers $a < b$, there is a rational number between a and b .
There is an irrational number between a and b .

4 Least upper bounds, greatest lower bounds (2.6)

Theorem 21. If S is a set of real numbers which is not empty and which has an upper bound, then it has a least upper bound.

Theorem 22. The greatest lower bound is unique whenever it exists.

Theorem 23. If S is a set of real numbers which is not empty and which has a lower bound, then it has a greatest lower bound.

Theorem 24. Given two non empty subsets of \mathbb{R} , A and B which have two lower bounds. If $A + B$ is defined by

$$A + B = \{a + b, a \in A, b \in B\}$$

then,

$$\inf(A + B) = \inf(A) + \inf(B)$$

Similar results holds for least upper bounds.

Proposition 25. If A and B are 2 non empty subsets of \mathbb{R} ,

1. If $-A$ is the set of all additive inverses of elements of A , then $\sup(-A) = -\inf(A)$.
2. If A is a subset of B , then $\inf(B) \leq \inf(A) \leq \sup(A) \leq \sup(B)$.

5 Limits of sequences

Definition 26. A sequence of real number is a function from \mathbb{N} to \mathbb{R} whose domain is \mathbb{N} .

Definition 27. Given a sequence of real number s_n , we say that $\lim_{n \rightarrow \infty} s_n = l$ if for each positive number ϵ , there is some integer N (depending on ϵ) such that if $n > N$, then $|s_n - l| < \epsilon$.

In this case we say that s_n is convergent.

We say that $\lim_{n \rightarrow \infty} s_n = \infty$ is for any number A , there is some integer N (depending on A) such that if $n > N$, then $s_n > A$.

Proposition 28.

$$\lim_{n \rightarrow \infty} \left(\frac{1}{n}\right) = 0$$

$$\lim_{n \rightarrow \infty} a^n = 0 \quad \text{if } 0 < a < 1$$

$$\lim_{n \rightarrow \infty} a^n = \infty \quad \text{if } 1 < a$$

Theorem 29. Given two convergent sequences s_n and t_n , then

- For any real numbers a and b , then

$$\lim_{n \rightarrow \infty} (as_n + bt_n) = a \left(\lim_{s \rightarrow \infty} s_n \right) + b \left(\lim_{s \rightarrow \infty} t_n \right).$$

- $\lim_{n \rightarrow \infty} (s_n t_n) = \left(\lim_{n \rightarrow \infty} s_n \right) \left(\lim_{n \rightarrow \infty} t_n \right).$

- If $\left(\lim_{n \rightarrow \infty} t_n \right) \neq 0$, then $\lim_{n \rightarrow \infty} \left(\frac{s_n}{t_n} \right) = \frac{\lim_{n \rightarrow \infty} s_n}{\lim_{n \rightarrow \infty} t_n}.$

Theorem 30. Given a sequence of real number s_n ,

- If, for any $n > N$ for some natural number N , $s_{n+1} \leq s_n$ and if there exists a real M such that $s_n \geq M$ for any n then s_n is convergent and $\lim_{n \rightarrow \infty} s_n \geq M.$
- If, for any $n > N$ for some natural number N , $s_{n+1} \geq s_n$ and if there exists a real M such that $s_n \leq M$ for any n then s_n is convergent and $\lim_{n \rightarrow \infty} s_n \leq M.$

Theorem 31. Squeeze theorem Given 3 sequences s_n , t_n and v_n , if there exists a natural number N

- for any $n > N$, $s_n \leq t_n \leq v_n$
- $\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} v_n = l$

Then t_n is convergent and $\lim_{n \rightarrow \infty} t_n = l.$

Theorem 32. Given 2 sequences s_n and t_n , assume $s_n \leq t_n$ for any $n > N$ for some N

- If $\lim_{n \rightarrow \infty} s_n = \infty$ then $\lim_{n \rightarrow \infty} t_n = \infty.$
- If $\lim_{n \rightarrow \infty} t_n = -\infty$, then $\lim_{n \rightarrow \infty} s_n = -\infty.$

Theorem 33. Nested Intervals (2.8) Given a sequence of nested closed intervals $I_n = [a_n, b_n]$, ($a_n \leq a_{n+1}$, and $b_n \geq b_{n+1}$), their intersection is either a singleton or a closed interval.