University of Washington

Last Name (PRINT): ____________________
First Name (PRINT): ____________________

Spring 2014 – Introductory Real Analysis
First Examination

Instructions

1. The use of all electronic devices is prohibited.
2. Present your solutions in the space provided. Show all your work neatly and concisely. Clearly indicate your final answer. You will be graded not merely on the final answer, but also on the quality and correctness of the work leading up to it.

Scholastic dishonesty will not be tolerated.
The work on this test is my own.

Signature:____________________

Questions	1	2	3	4	5	Total
Grade:						

Exercise 1. (6 points) Using the axioms of $\mathbb{R}$

1. Prove that the additive inverse is unique.

2. Prove that for any real numbers x and y, $(-x)y = x(-y) = -(xy)$. *In addition to the axioms of* $\mathbb{R}$*, you may use without proof the properties 1-4 from Proposition 2.*

Exercise 2. (4 points) Prove that if a real number x satisfies $|x - 3| < 2$, then $1 < x < 5$.

Exercise 3. (5 points) Find the least upper bound and the greatest lower bound of

$$A = \left\{ \frac{(3n+1)}{n+1} \text{ for all natural number } n \right\}$$

Exercise 4. (5 points) Let s_n and t_n be two sequences such that

$$\text{for any natural number } n, \quad s_n > t_n \qquad \text{and} \qquad \lim_{n\to\infty} t_n = \infty,$$

Prove using the definition of the limits that $\lim_{n\to\infty} s_n = \infty$. (Proposition 32)

Exercise 5. (5 points) Using the definition of the limit, prove that

$$\lim_{n\to\infty} \frac{(n+1)^2}{n^2} = 1.$$

1 Axioms of $\mathbb{R}$

Axiom 1. $\mathbb{R}$ is a commutative field.

Definition 1. $F + *$ is a commutative field if

1. For any a, b, c in F, $a+(b+c) = (a+b)+c$, and $a*(b*c) = (a*b)*c$ (associativity).
2. For any a and b in F, $a + b = b + a$ and $a * b = b * a$ (commutativity).
3. For any a, b, c in F, $a*(b+c) = a*b+a*c$ (distributivity).
4. There exists an element written 0 such that for any a in F, $a + 0 = a$ (additive identity)
5. There exists an element written 1 such that for any a in F, $a * 1 = a$ (multiplicative identity)
6. For any a in F, there exists an element written $-a$ such that $a + (-a) = 0$ (additive inverse)
7. For any a in F except 0, there exists an element written a^{-1} such that $a * (a^{-1}) = 1$ (multiplicative inverse).

Proposition 2.

1. The additive inverse is unique.
2. For any real number a, $0 * a = 0$.
3. $(-1)(-1) = 1$.
4. The additive inverse of $a + b$ is $(-a) + (-b)$.
5. for any real number a, $-a = (-1) * a$.

Axiom 2. $\mathbb{R}$ is totally ordered:
There is a relation $>$ that satisfies

1. If $a > 0$ and $b > 0$, then $a + b > 0$
2. If $a > 0$ and $b > 0$, then $a * b > 0$
3. For each a only one of the following is true
   (a) $a > 0$,
   (b) $a = 0$,
   (c) $-a > 0$

Definition 3. We say that $a < b$ if $b - a > 0$

Proposition 4. Given three real numbers a, b, c,

1. $0 < a^2$ if $a \neq 0$
2. If $a < b$ and $b < c$ then $a < c$
3. If $a < b$ and $0 < c$, then $ac < bc$
4. If $a < b$, for any c, $a + c < b + c$

Proposition 5.

1. If a is a positive real number, then its multiplicative inverse a^{-1} is positive.
2. For any real numbers such that $0 < a < b$, then $b^{-1} < a^{-1}$.

Definition 6. The absolute value of a, written $|a|$ is

- $|a| = a$ if $a > 0$
- $|a| = -a$ if $a < 0$
- $|a| = 0$ if $a = 0$

Proposition 7. For any a, b in $\mathbb{R}$,

- $|a * b| = |a| * |b|$
- $|a + b| \leqslant |a| + |b|$
- $||a| - |b|| \leqslant |a - b|$

Proposition 8. For any real numbers such that $a < b < c$, $|b| \leqslant max\left(|a|, |c|\right)$.

Axiom 3. (Axiom of continuity) Suppose that all real numbers are separated into two collections which we denote by L and R, in such a way that

1. Every number is either in L or in R.
2. Each collection contains at least 1 element.
3. If a is in L and b is in R, then $a < b$.

then there is a number c in $\mathbb{R}$ such that all numbers less than c are in L and all numbers greater that c are in R.

Proposition 9. The cut number c is unique.

Theorem 10. (Archimedian law of real numbers) Let a and b be 2 positive real numbers. There exists a positive integer n such that $b < na$.

Proposition 11. If a real number y satisfies that $0 \leqslant y \leqslant \frac{1}{n}$ for any natural number n, then $y = 0$.

Proposition 12. For any positive rel number y, there exists a real number c such that $c^2 = y$.

Theorem 13. Any system that satisfies Axiom 1, 2, and 3 is isomorphic to $\mathbb{R}$

2 Axioms of $\mathbb{N}$ (2.3)

Axiom 4. The natural numbers are the smallest class of real numbers that satisfies

1. 1 is an element of $\mathbb{N}$
2. If n is an element of $\mathbb{N}$, $n + 1$ is an element of $\mathbb{N}$

Proposition 14. Any natural number either equals to 1 or is greater than 1.

Proposition 15. For any 2 natural numbers $p < n$, there exist a natural number m such that $n = p + m$.

Proposition 16. Let n be a natural number. there is no natural number between n and $n + 1$.

Proposition 17. If S is a class of positive integers containing at least 1 element, it contains a smallest element.

3 Rational & irrational numbers (2.5)

Definition 18. A rational number is a number that can be written in the form $\pm\frac{p}{q}$, where p and q are 2 natural numbers.
An irrational number is a number that is not rational

Theorem 19. The number $\sqrt{2}$ is irrational.

Theorem 20. Given to numbers $a < b$, there is a rational number between a and b.
There is an irrational number between a and b.

4 Least upper bounds, greatest lower bounds (2.6)

Theorem 21. If S is a set of real numbers which is not empty and which has an upper bound, then it has a least upper bound.

Exercise 6. Given the set $E = \{1 + \frac{1}{n}, \text{ for } n \in \mathbb{N}^*\}$

- Find an upper bound, and a lower bound.
- What is the least upper bound, the greatest lower bound?

Theorem 22. The greatest lower bound is unique whenever it exists.

Theorem 23. If S is a set of real numbers which is not empty and which has a lower bound, then it has a greatest lower bound.

Theorem 24. Given two non empty subsets of $\mathbb{R}$, A and B which have two lower bounds. If $A + B$ is defined by

$$A + B = \{a + b, a \in A, b \in B\}$$

then,

$$inf(A + B) = inf(A) + inf(B)$$

Similar results holds for least upper bounds.

Proposition 25. If A and B are 2 non empty subsets of $\mathbb{R}$,

1. If $-A$ is the set of all additive inverses of elements of A, then $\sup(-A) = -\inf(A)$.
2. If A is a subset of B, then $\inf(B) \leqslant \inf(A) \leqslant \sup(A) \leqslant \sup(B)$.

5 Limits of sequences

Definition 26. A sequence of real number is a function from $\mathbb{N}$ to $\mathbb{R}$ whose domain is $\mathbb{N}$.

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Definition 27. Given a sequence of real number s_n, we say that $\lim_{n\to\infty} s_n = l$ if for each positive number ϵ, there is some integer N (depending on ϵ) such that if $n > N$, then $|s_n - l| < \epsilon$.
In this case we say that s_n is convergent.
We say that $\lim_{n\to\infty} s_n = \infty$ is for any number A, there is some integer N (depending on A) such that if $n > N$, then $s_n > A$.

Proposition 28.

$$\lim_{n\to\infty}\left(\frac{1}{n}\right)=0$$

$$\lim_{n\to\infty} a^n = 0 \qquad \text{if} \quad 0<a<1$$

$$\lim_{n\to\infty} a^n = \infty \qquad \text{if} \quad 1<a$$

Theorem 29. Given two convergent sequences s_n and t_n, then

- For any real numbers a and b, then
$$\lim_{n\to\infty}(as_n+bt_n)=a\left(\lim_{s\to\infty}s_n\right)+b\left(\lim_{s\to\infty}t_n\right).$$
- $\lim_{n\to\infty}(s_nt_n)=\left(\lim_{n\to\infty}s_n\right)\left(\lim_{n\to\infty}t_n\right)$.
- If $\left(\lim_{n\to\infty}t_n\right)\neq 0$, then $\lim_{n\to\infty}\left(\frac{s_n}{t_n}\right)=\frac{\lim_{n\to\infty}s_n}{\lim_{n\to\infty}t_n}$.

Theorem 30. Given a sequence of real number s_n,

- If, for any $n>N$ for some natural number N, $s_{n+1}\leqslant s_n$ and if there exists a real M such that $s_n\geqslant M$ for any n then s_n is convergent and $\lim_{n\to\infty}s_n\geqslant M$.
- If, for any $n>N$ for some natural number N, $s_{n+1}\geqslant s_n$ and if there exists a real M such that $s_n\leqslant M$ for any n then s_n is convergent and $\lim_{n\to\infty}s_n\leqslant M$.

Theorem 31. Squeeze theorem Given 3 sequences s_n, t_n and v_n, if there exists a natural number N

- for any $n>N$, $s_n\leqslant t_n\leqslant v_n$
- $\lim_{n\to\infty}s_n=\lim_{n\to\infty}v_n=l$

Then t_n is convergent and $\lim_{n\to\infty}t_n=l$.

Theorem 32. Given 2 sequences s_n and t_n, assume $s_n \leqslant t_n$ for any $n > N$ for some N

- If $\lim\limits_{n\to\infty} s_n = \infty$ then $\lim\limits_{n\to\infty} t_n = \infty$.
- If $\lim\limits_{n\to\infty} t_n = -\infty$, then $\lim\limits_{n\to\infty} s_n = -\infty$.