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Last Name (PRINT): ____________________
First Name (PRINT): ____________________

Fall 2014 – Introductory Real Analysis
Second Examination

Instructions

1. The use of all electronic devices is prohibited.
2. Present your solutions in the space provided. Show all your work neatly and concisely. Clearly indicate your final answer. You will be graded not merely on the final answer, but also on the quality and correctness of the work leading up to it.

Scholastic dishonesty will not be tolerated.
The work on this test is my own.

Signature:________________________

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Grade:

Exercise 1. (5 points) Is the set of rational numbers $\mathbb{Q}$ open?

Is the set $\mathbb{Q}$ closed?

Justify your answers.

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Exercise 2. (5 points) Let S be a bounded set and let $inf(S)$ be its greatest lower bound. Assume that $inf(S)$ is not an element of S.

Prove that $inf(S)$ is an accumulation point of S.

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Exercise 3. (4 points) Find the accumulation points of $S = \left\{ \dfrac{(-1)^n n}{2n+5} \right\}$. Justify your answer.

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Exercise 4. (5 points) Given a series with positive terms $\sum_{n=1}^{\infty} a_n$, prove that if the partial sum $S_n = \sum_{k=1}^{n} a_k$ are bounded, then the series is convergent.

It is part of theorem 17.You may use any result mentioned prior to Theorem 17.

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Exercise 5. (6 points) Determine if the following series are convergent, divergent, conditionally convergent, or absolutely convergent. Justify your answer. You will be graded merely on the justification you provide.

1. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3n}{n^2+1}\right)^3.$

2. $\displaystyle\sum_{n=1}^{\infty}\frac{\sin n}{n^2}.$

3. $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n}{n+1}.$

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1 Section 1.62

Definition 1. A sequence of real numbers is a function from $\mathbb{N}$ to $\mathbb{R}$ whose domain is $\mathbb{N}$.

Definition 2. Given a sequence of real number s_n, we say that $\lim_{n\to\infty} s_n = l$ if for each positive number ϵ, there is some integer N (depending on ϵ) such that if $n > N$, then $|s_n - l| < \epsilon$.
In this case we say that s_n is convergent.
We say that $\lim_{n\to\infty} s_n = \infty$ is for any number A, there is some integer N (depending on A) such that if $n > N$, then $s_n > A$.

Proposition 3.

$$\lim_{n\to\infty} \left(\frac{1}{n}\right) = 0$$

$$\lim_{n\to\infty} a^n = 0 \qquad \text{if} \quad 0 < a < 1$$

$$\lim_{n\to\infty} a^n = \infty \qquad \text{if} \quad 1 < a$$

Theorem 4. Given two convergent sequences s_n and t_n, then

- For any real numbers a and b, then
$$\lim_{n\to\infty} (as_n + bt_n) = a\left(\lim_{s\to\infty} s_n\right) + b\left(\lim_{s\to\infty} t_n\right).$$
- $\lim_{n\to\infty} (s_n t_n) = \left(\lim_{n\to\infty} s_n\right)\left(\lim_{n\to\infty} t_n\right)$.
- If $\left(\lim_{n\to\infty} t_n\right) \neq 0$, then $\lim_{n\to\infty} \left(\frac{s_n}{t_n}\right) = \frac{\lim_{n\to\infty} s_n}{\lim_{n\to\infty} t_n}$.

2 Chapter 16

2.1 Open sets, Closed sets

Definition 5. A neighborhood of x_0 is a set of points x such that $|x - x_0| < h$ or equivalently $(x_0 - h, x_0 + h)$ for some positive real h.

Definition 6. A set S is open if any element s in S has a neighborhood $(s - h, s + h)$ entirely included in S.
A set S is closed if its complementary is open.

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Remark: the sets

- $\emptyset$ and $(-\infty, \infty)$ are both open and closed.
- (a, b) is open with a being possibly $-\infty$ and b being ∞.
- $\mathbb{R} \setminus \{a\}$ is open for any real a.
- $[a, b]$, $(-\infty, b]$, $[a, \infty)$, $\{a\}$ are closed.
- The set $\mathbb{R} \setminus \left\{1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots\right\}$ is neither closed nor open.

Theorem 7. If A and B are two open sets, then

- $A \cup B$ is open
- $A \cap B$ is open.

If A and B are closed sets, then

- $A \cup B$ is closed,
- $A \cap B$ is closed.

2.2 Accumulation points

Definition 8. A element y is an accumulation point for a set S if any neighborhood of y contains at least one element of S that is not y. i. e for any h the intersection $S \cap (y-h, y+h) \setminus \{y\}$ is not empty.

Theorem 9. A set S is closed if and only if any accumulation point of S is in S.

Theorem 10. (Bolzano Weierstrass) Any infinite bounded set has at least one accumulation point.

Theorem 11. An element y is an accumulation of a set S iff there exists a sequence of distinct elements of S that is convergent to y.

Theorem 12. Given a convergent sequence s_n of real numbers that takes infinitely many distinct values, the set $S = \{s_n, n \in \mathbb{N}\}$ has exactly one accumulation point, the limit of s_n.

Theorem 13. Let s_n be a bounded sequence, s_n has a convergent subsequence.

2.3 Cauchy sequences

Definition 14. s_n is a Cauchy sequence if for any positive real ϵ, there exist an integer N such that for any $p > N$ and $q > N$, then $|s_p - s_q| < \epsilon$

Theorem 15. A sequence s_n is convergent if and only if s_n is a Cauchy sequence.

3 Chapter 19

Theorem 16. If a series $\sum a_n$ is convergent then $\lim_{n\to\infty} a_n = 0$.

3.1 Section of nonnegative terms

Remark: The series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

Theorem 17. Suppose that $u_n \geqslant 0$ for every n, then the series $\sum u_n$ is convergent iff $\sum_{k=1}^{n} u_k$ is bounded.

Theorem 18. Let $\sum a_n$ and $\sum b_n$ series with non negative terms such that for all the index greater than some N, $0 \leqslant a_n \leqslant b_n$.

- If $\sum b_n$ is convergent, then $\sum a_n$ is convergent.
- If $\sum a_n$ is divergent, then $\sum b_n$ is divergent.

Theorem 19. Let $\sum a_n$ and $\sum b_n$ series with positive terms such that $\lim_{n\to\infty} \frac{a_n}{b_n} = c \neq 0$, then either $\sum a_n$ and $\sum b_n$ are both convergent or $\sum a_n$ and $\sum b_n$ are both divergent.

Theorem 20. (Ratio Test) Let $\sum a_n$ and $\sum b_n$ series with positive terms such that $\frac{a_{n+1}}{a_n} \leqslant \frac{b_{n+1}}{b_n}$ for any n, then

- If $\sum b_n$ is convergent, then $\sum a_n$ is convergent.
- If $\sum a_n$ is divergent, then $\sum b_n$ is divergent.

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Theorem 21.

- If $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = r < 1$ then $\sum u_n$ is convergent.
- If $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = r > 1$ then $\sum u_n$ is divergent.

Theorem 22. (Comparison to a geometric series) If for any n, $\frac{u_{n+1}}{u_n} < r < 1$, then $\sum u_n$ is convergent.

Theorem 23. (comparison with an integral) Given a positive function f that is non increasing on a interval $[1, \infty)$. Then the series $\sum f(n)$ and $\int_1^\infty f(t)\mathrm{d}t$ are both convergent or they are both divergent.

3.2 Absolute convergence and conditional convergence

Definition 24. Given a series $\sum u_n$, $\sum u_n$ is absolutely convergent if $\sum |u_n|$ is convergent.

Theorem 25. If $\sum u_n$ is absolutely convergent, then $\sum u_n$ is convergent.

Theorem 26. Given a series $\sum u_n$, if a_n is the subsequence of u_n corresponding to the positive terms.If b_n is the subsequence of u_n corresponding to the non positive terms (possibly completed by 0 terms is one of the sequence is finite)

- If $\sum u_n$ is absolutely convergent, then $\sum a_n$ and $\sum b_n$ are convergent and $\sum u_n = \sum a_n + \sum b_n$.
- If $\sum u_n$ is conditionally convergent, then both $\sum a_n$ and $\sum b_n$ are divergent.

Definition 27. If $\sum u_n$ is convergent and not absolutely convergent, then $\sum u_n$ is conditionally convergent.

Theorem 28. (Alternating series) A series $\sum u_n$ such that its terms alternate between positive and negative, and such that $|u_n|$ is decreasing toward 0 is convergent.