University of Washington

Last Name (PRINT): ________________
First Name (PRINT): ________________

Summer 2014 – Introductory Real Analysis
Second Examination

Instructions

1. The use of all electronic devices is prohibited.
2. Present your solutions in the space provided. Show all your work neatly and concisely. Clearly indicate your final answer. You will be graded not merely on the final answer, but also on the quality and correctness of the work leading up to it.

Scholastic dishonesty will not be tolerated.
The work on this test is my own.

Signature:________________

Questions	1	2	3	4	5	6	Total
Grade:							

Exercise 1. (4 points) Given a collection of open sets S_k ($k \in \mathbb{N}$),

prove that for any $n \in \mathbb{N}$, the finite union $\bigcup_{k=1}^{n} S_k$ is open.

Let S be $S = \bigcup_{k=1}^{n} S_k$

$\forall x \in S$, $\exists k$ such that $x \in S_k$

S_k open $\Rightarrow$ there exists a neighborhood of x, $(x-h, x+h)$ entirely included in S_k

Since $S_k \subset S$, the neighborhood of S is entirely included in S

$\Rightarrow$ S is open

Exercise 2. (4 points) Given G a finite set of real numbers, prove that G is closed.

G is a Finite set $\Rightarrow G = \{a_1, a_2, \dots, a_n\}$ for some integer n.

Without loss of generality, we may assume that $a_1 < a_2 < a_3 \dots < a_n$

$G^c = (-\infty, a_1) \cup (a_1, a_2) \cup \dots \cup (a_{n-1}, a_n) \cup (a_n, \infty)$

G^c is a Finite union of open intervals.

$\Rightarrow G^c$ is open $\Rightarrow G$ is closed.

Exercise 3. (4 points) Prove that if S is closed, then each accumulation point of S is in S.
The result is part of theorem 9. You may use any material mentioned prior to theorem 9.

S is closed

Let y be an accumulation of S.

Let's prove the result by contradiction and assume that $y \notin S$.

$y \notin S \Rightarrow y \in S^c$

S is closed $\Rightarrow S^c$ is open

$\Rightarrow$ there exists a neighborhood of y entirely included in S^c: $(y-h, y+h)$

this neighborhood does not intersect $S \Rightarrow y$ cannot be an accumulation.

contradiction.
$\Rightarrow$ If y is an accumulation point of a closed set S, y is in S.

Exercise 4. (4 points) Show that if the sequence $(a_n)_{(n\in\mathbb{N})}$ is a Cauchy sequence, then the sequence $(a_n^2)_{(n\in\mathbb{N})}$ is also a Cauchy sequence.

If $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence then a_n is convergent.

Let L be the limit of a_n

a_n^2 is also a convergent sequence and $\lim_{n\to\infty} a_n^2 = L^2$ (rules about product of sequences)

a_n^2 is convergent $\Rightarrow$ a_n^2 is a Cauchy sequence.

Exercise 5. (4 points) Prove that if a series $\sum_{n=1}^{\infty} a_n$ is convergent then $\lim_{n\to\infty} a_n = 0$.

The result is theorem 16. You may use any material mentioned prior to theorem 16.

If $\sum_{n=1}^{\infty} a_n$ is convergent, then the partial sum $S_n = \sum_{k=1}^{n} a_n$ is a Cauchy sequence.

$\Rightarrow \forall \varepsilon > 0 \ \exists N$ such that $\forall n \forall p$: $n > p > N$, $|S_n - S_p| < \varepsilon$

For $p = n-1$, $S_n - S_p = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n-1} a_k = a_n$

Therefore $\forall \varepsilon > 0 \ \exists N > 0$ such that $n-1 > N$ then $|S_n - S_{n-1}| = |a_n| < \varepsilon$

$n > N+1$

Exercise 6. (5 points) Determine the convergent or divergence of the following series

1. $R = \sum_{n=-}^{\infty} \frac{n+5}{n^3+2n+1}$.
2. $S = \sum_{n=1}^{\infty} \frac{2^n}{n!}$.
3. $T = \sum_{n=1}^{\infty} \frac{n^2-1}{n^2+1}$.

1) Let $u_n = \frac{n+5}{n^3+2n+1}$ u_n is a positive term for any n.

$$\left.\begin{array}{l} n+5 < 6n \\ n^3+2n+1 > n^3 \end{array}\right\} \quad u_n < \frac{6n}{n^3} = \frac{6}{n^2}$$

by the comparison theorem. $6\sum \frac{1}{n^2}$ is convergent $\Rightarrow \sum u_n = R$ is also convergent.

2) $u_n = \frac{2^n}{n!}$ (positive term)

$$u_n = \frac{2}{1} \cdot \frac{2}{2} \cdot \frac{2}{3} \cdot \dots \frac{2}{(n-1)} \cdot \frac{2}{n} \leqslant \frac{8}{(n-1)(n)} \leqslant \frac{4}{n^2}$$

the series $\sum \frac{4}{n^2}$ is convergent $\Rightarrow \sum u_n$ is also convergent.

Remark. The ratio test would be more appropriate for this series:

$$\frac{u_{n+1}}{u_n} = \frac{2}{n+1} \xrightarrow[n\to\infty]{} 0$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} < 1 \Rightarrow \sum u_n \text{ is convergent.}$$

3) $t_n = \frac{n^2-1}{n^2+1}$ $\lim_{n\to\infty} t_n = 1 \neq 0 \Rightarrow \sum t_n$ is divergent. (contrapositive of 1st theorem about series)