University of Washington

Last Name (PRINT): ______________________
First Name (PRINT): ______________________

Spring 2014 – Introductory Real Analysis
Final Examination

Instructions

1. The use of all electronic devices is prohibited.
2. Present your solutions in the space provided. Show all your work neatly and concisely. Clearly indicate your final answer. You will be graded not merely on the final answer, but also on the quality and correctness of the work leading up to it.

Scholastic dishonesty will not be tolerated.
The work on this test is my own.

Signature:________________________

Questions	1	2	3	4	5	6	7	8	9	Total
Grade:										

Exercise 1. (12 point) Prove the following proposition:

- The additive inverse is unique.

  This result is Proposition 2.1. You may use for the proof any results mentioned PRIOR to Proposition 2.1

- For any real number a, $-a = (-1) * a$.

  This result is Proposition 2.5. You may use for the proof any results mentioned PRIOR to Proposition 2.5

Exercise 2. (10 points) Prove proposition 9: The cut number is unique.

You may use any result prior to Proposition 9.

You may try a proof by contradiction and assuming that there exists 2 different cut numbers.

Exercise 3. (12 points) Prove that if A is a non empty subset of B,

$$\inf(B) \leqslant \inf(A) \leqslant \sup(A) \leqslant \sup(B).$$

You may use any result prior to Proposition 25.

Exercise 4. (10 points) Use the definition of limits (Definition 27 with ϵ) to prove that

$$\lim_{n\to\infty} \frac{n\sin n}{n^2+1} = 0$$

Exercise 5. (12 points) Let S be the set of rational numbers r such that $0 < r < 1$, is the set S open? is it closed? Justify your answer.

Exercise 6. (8 points) Let S be a non empty subset of the real numbers. Assume that L is the greatest lower bound of S and L is not in S,

Prove that L is an accumulation point of S.

Exercise 7. (14 points) Determine whether the following series are convergent, absolutely convergent, conditionally convergent.

1. $$S = \sum_{n=1}^{\infty} \frac{(n+1)(n+2)}{2^n n^2}$$

2. $$T = \sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{1+n}$$

Exercise 8. (15 points) Given the sequence of functions

$$f_n(x) = \frac{n + \cos(nx)}{2n + 1}$$

1. Is f_n convergent?

2. Is f_n uniformly convergent on $\mathbb{R}$? Justify your answer.

3. Let $f(x)$ be the limit $\lim_{n\to\infty} f_n(x)$, whenever it exists. Is $f'(x) = \lim_{n\to\infty} f_n'(x)$? Explain why it does not contradict Theorem 62.

Exercise 9. (8 points) Consider the series

$$S(x) = \sum_{n=1}^{\infty} \frac{1}{n^2 + x^2}$$

Show that the series converges uniformly on $\mathbb{R}$.

1 Axioms of $\mathbb{R}$

Axiom 1. $\mathbb{R}$ is a commutative field.

Definition 1. $F + *$ is a commutative field if

1. For any a, b, c in F, $a+(b+c) = (a+b)+c$, and $a*(b*c) = (a*b)*c$ (associativity).
2. For any a and b in F, $a + b = b + a$ and $a * b = b * a$ (commutativity).
3. For any a, b, c in F, $a*(b+c) = a*b+a*c$ (distributivity).
4. There exists an element written 0 such that for any a in F, $a + 0 = a$ (additive identity)
5. There exists an element written 1 such that for any a in F, $a * 1 = a$ (multiplicative identity)
6. For any a in F, there exists an element written $-a$ such that $a + (-a) = 0$ (additive inverse)
7. For any a in F except 0, there exists an element written a^{-1} such that $a * (a^{-1}) = 1$ (multiplicative inverse).

Proposition 2. 1. The additive inverse is unique.

2. For any real number a, $0 * a = 0$.
3. $(-1)(-1) = 1$.
4. The additive inverse of $a + b$ is $(-a) + (-b)$.
5. for any real number a, $-a = (-1) * a$.

Axiom 2. $\mathbb{R}$ is totally ordered:
There is a relation $>$ that satisfies

1. If $a > 0$ and $b > 0$, then $a + b > 0$
2. If $a > 0$ and $b > 0$, then $a * b > 0$
3. For each a only one of the following is true
   (a) $a > 0$,
   (b) $a = 0$,
   (c) $-a > 0$

Definition 3. We say that $a < b$ if $b - a > 0$

Proposition 4. Given three real numbers a, b, c,

1. $0 < a^2$ if $a \neq 0$
2. If $a < b$ and $b < c$ then $a < c$
3. If $a < b$ and $0 < c$, then $ac < bc$
4. If $a < b$, for any c, $a + c < b + c$

Proposition 5.
1. If a is a positive real number, then its multiplicative inverse a^{-1} is positive.
2. For any real numbers such that $0 < a < b$, then $b^{-1} < a^{-1}$.

Definition 6. The absolute value of a, written $|a|$ is

- $|a| = a$ if $a > 0$
- $|a| = -a$ if $a < 0$
- $|a| = 0$ if $a = 0$

Proposition 7. For any a, b in $\mathbb{R}$,

- $|a * b| = |a| * |b|$
- $|a + b| \leqslant |a| + |b|$
- $||a| - |b|| \leqslant |a - b|$

Proposition 8. For any real numbers such that $a < b < c$, $|b| \leqslant max\left(|a|, |c|\right)$.

Axiom 3. (Axiom of continuity) Suppose that all real numbers are separated into two collections which we denote by L and R, in such a way that

1. Every number is either in L or in R.
2. Each collection contains at least 1 element.
3. If a is in L and b is in R, then $a < b$.

then there is a number c in $\mathbb{R}$ such that all numbers less than c are in L and all numbers greater that c are in R.

Proposition 9. The cut number c is unique.

Theorem 10. (Archimedian law of real numbers) Let a and b be 2 positive real numbers. There exists a positive integer n such that $b < na$.

Proposition 11. If a real number y satisfies that $0 \leqslant y \leqslant \frac{1}{n}$ for any natural number n, then $y = 0$.

Proposition 12. For any positive rel number y, there exists a real number c such that $c^2 = y$.

Theorem 13. Any system that satisfies Axiom 1, 2, and 3 is isomorphic to $\mathbb{R}$

2 Axioms of $\mathbb{N}$ (2.3)

Axiom 4. The natural numbers are the smallest class of real numbers that satisfies

1. 1 is an element of $\mathbb{N}$
2. If n is an element of $\mathbb{N}$, $n + 1$ is an element of $\mathbb{N}$

Proposition 14. Any natural number either equals to 1 or is greater than 1.

Proposition 15. For any 2 natural numbers $p < n$, there exist a natural number m such that $n = p + m$.

Proposition 16. Let n be a natural number. there is no natural number between n and $n + 1$.

Proposition 17. If S is a class of positive integers containing at least 1 element, it contains a smallest element.

3 Rational & irrational numbers (2.5)

Definition 18. A rational number is a number that can be written in the form $\pm\frac{p}{q}$, where p and q are 2 natural numbers.
An irrational number is a number that is not rational

Theorem 19. The number $\sqrt{2}$ is irrational.

Theorem 20. Given to numbers $a < b$, there is a rational number between a and b.
There is an irrational number between a and b.

4 Least upper bounds, greatest lower bounds (2.6)

Theorem 21. If S is a set of real numbers which is not empty and which has an upper bound, then it has a least upper bound.

Exercise 10. Given the set $E = \{1 + \frac{1}{n}, \text{ for } n \in \mathbb{N}^*\}$

- Find an upper bound, and a lower bound.
- What is the least upper bound, the greatest lower bound?

Theorem 22. The greatest lower bound is unique whenever it exists.

Theorem 23. If S is a set of real numbers which is not empty and which has a lower bound, then it has a greatest lower bound.

Theorem 24. Given two non empty subsets of $\mathbb{R}$, A and B which have two lower bounds. If $A + B$ is defined by

$$A + B = \{a + b, a \in A, b \in B\}$$

then,

$$inf(A + B) = inf(A) + inf(B)$$

Similar results holds for least upper bounds.

Proposition 25. If A and B are 2 non empty subsets of $\mathbb{R}$,

1. If $-A$ is the set of all additive inverses of elements of A, then $\sup(-A) = -\inf(A)$.
2. If A is a subset of B, then $\inf(B) \leqslant \inf(A) \leqslant \sup(A) \leqslant \sup(B)$.

5 Limits of sequences

Definition 26. A sequence of real number is a function from $\mathbb{N}$ to $\mathbb{R}$ whose domain is $\mathbb{N}$.

Definition 27. Given a sequence of real number s_n, we say that $\lim_{n\to\infty} s_n = l$ if for each positive number ϵ, there is some integer N (depending on ϵ) such that if $n > N$, then $|s_n - l| < \epsilon$.
In this case we say that s_n is convergent.
We say that $\lim_{n\to\infty} s_n = \infty$ is for any number A, there is some integer N (depending on A) such that if $n > N$, then $s_n > A$.

Proposition 28.

$$\lim_{n\to\infty}\left(\frac{1}{n}\right)=0$$

$$\lim_{n\to\infty} a^n = 0 \qquad \text{if} \quad 0<a<1$$
$$\lim_{n\to\infty} a^n = \infty \qquad \text{if} \quad 1<a$$

Theorem 29. Given two convergent sequences s_n and t_n, then

- For any real numbers a and b, then

$$\lim_{n\to\infty}(as_n+bt_n)=a\left(\lim_{s\to\infty}s_n\right)+b\left(\lim_{s\to\infty}t_n\right).$$

- $\lim_{n\to\infty}(s_nt_n)=\left(\lim_{n\to\infty}s_n\right)\left(\lim_{n\to\infty}t_n\right)$.
- If $\left(\lim_{n\to\infty}t_n\right)\neq 0$, then $\lim_{n\to\infty}\left(\frac{s_n}{t_n}\right)=\frac{\lim_{n\to\infty}s_n}{\lim_{n\to\infty}t_n}$.

Theorem 30. Given a sequence of real number s_n,

- If, for any $n>N$ for some natural number N, $s_{n+1}\leqslant s_n$ and if there exists a real M such that $s_n\geqslant M$ for any n then s_n is convergent and $\lim_{n\to\infty}s_n\geqslant M$.
- If, for any $n>N$ for some natural number N, $s_{n+1}\geqslant s_n$ and if there exists a real M such that $s_n\leqslant M$ for any n then s_n is convergent and $\lim_{n\to\infty}s_n\leqslant M$.

Theorem 31. Squeeze theorem Given 3 sequences s_n, t_n and v_n, if there exists a natural number N

- for any $n>N$, $s_n\leqslant t_n\leqslant v_n$
- $\lim_{n\to\infty}s_n=\lim_{n\to\infty}v_n=l$

Then t_n is convergent and $\lim_{n\to\infty}t_n=l$.

Theorem 32. Given 2 sequences s_n and t_n, assume $s_n \leqslant t_n$ for any $n > N$ for some N

- If $\lim_{n\to\infty} s_n = \infty$ then $\lim_{n\to\infty} t_n = \infty$.
- If $\lim_{n\to\infty} t_n = -\infty$, then $\lim_{n\to\infty} s_n = -\infty$.

Theorem 33. (Nested intervals) Given a sequence of nested intervals $I_n = [a_n, b_n]$, with $a_n \leqslant a_{n+1} \leqslant b_{n+1} \leqslant b_n$, their intersection is either a singleton or a closed interval.

6 Chapter 16

6.1 Open sets, Closed sets

Definition 34. A neighborhood of x_0 is a set of points x such that $|x - x_0| < h$ or equivalently $(x_0 - h, x_0 + h)$ for some positive real h.

Definition 35. A set S is open if any element s in S has a neighborhood $(s - h, s + h)$ entirely included in S.
A set S is closed if its complementary is open.

Remark: the sets

- $\emptyset$ and $(-\infty, \infty)$ are both open and closed.
- (a, b) is open with a being possibly $-\infty$ and b being ∞.
- $\mathbb{R} \setminus \{a\}$ is open for any real a.
- $[a, b]$, $(-\infty, b]$, $[a, \infty)$, $\{a\}$ are closed.
- The set $\mathbb{R} \setminus \left\{1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots\right\}$ is neither closed nor open.

Theorem 36. If A and B are two open sets, then

- $A \cup B$ is open
- $A \cap B$ is open.

If A and B are closed sets, then

- $A \cup B$ is closed,
- $A \cap B$ is closed.

6.2 Accumulation points

Definition 37. A element y is an accumulation point for a set S if any neighborhood of y contains at least one element of S that is not y. i. e for any h the intersection $S \cap (y-h, y+h) \setminus \{y\}$ is not empty.

The only accumulation point of the set $S = \left\{ \frac{1}{n}, n \in \mathbb{N} \right\}$ is 0.

Theorem 38. A set S is closed if and only if any accumulation point of S is in S.

Theorem 39. (Bolzano Weierstrass) Any infinite bounded set has at least one accumulation point.

Theorem 40. Given a convergent sequence s_n of real numbers that takes infinitely many distinct values, the set $S = \{s_n, n \in \mathbb{N}\}$ has exactly one accumulation point.

Theorem 41. If S be a set with an accumulation point y, then there exists a sequence of elements of S that converges to y.

Theorem 42. Let s_n be a bounded sequence, s_n has a convergent subsequence.

6.3 Cauchy sequences

Definition 43. s_n is a Cauchy sequence if for any positive real ϵ, there exist an integer N such that for any $p > N$ and $q > N$, then $|s_p - s_q| < \epsilon$

Theorem 44. A sequence s_n is convergent if and only if s_n is a Cauchy sequence.

7 Chapter 19

Theorem 45. If a series $\sum a_n$ is convergent then $\lim_{n\to\infty} a_n = 0$.

Remark: The converse is false. The series $\sum \frac{1}{n}$ is divergent and $\lim n \to \infty \frac{1}{n} = 0$.

7.1 Section of nonnegative terms

Remark:

- The series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.
- The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

Theorem 46. Suppose that $u_n \geqslant 0$ for every n, then the series $\sum u_n$ is convergent iff $\sum_{k=1}^{n} u_k$ is bounded.

Theorem 47. Let $\sum a_n$ and $\sum b_n$ series with non negative terms such that for all the index greater than some N, $0 \leqslant a_n \leqslant b_n$.

- If $\sum b_n$ is convergent, then $\sum a_n$ is convergent.
- If $\sum a_n$ is divergent, then $\sum b_n$ is divergent.

Theorem 48. Let $\sum a_n$ and $\sum b_n$ series with positive terms such that $\lim_{n\to\infty} \frac{a_n}{b_n} = c \neq 0$, then either $\sum a_n$ and $\sum b_n$ are both convergent or $\sum a_n$ and $\sum b_n$ are both divergent.

Theorem 49. (Ratio Test) Let $\sum a_n$ and $\sum b_n$ series with positive terms such that $\frac{a_{n+1}}{a_n} \leqslant \frac{b_{n+1}}{b_n}$ for any n, then

- If $\sum b_n$ is convergent, then $\sum a_n$ is convergent.
- If $\sum a_n$ is divergent, then $\sum b_n$ is divergent.

Theorem 50. (Comparison to a geometric series) If for any n, $\frac{u_{n+1}}{u_n} < r < 1$, then $\sum u_n$ is convergent.

Theorem 51. (comparison with an integral) Given a positive function f that is non increasing on a interval $[1, \infty)$. Then the series $\sum f(n)$ and $\int_1^{\infty} f(t)\mathrm{d}t$ are both convergent or they are both divergent.

7.2 Absolute convergence and conditional convergence

Definition 52. Given a series $\sum u_n$, $\sum u_n$ is absolutely convergent if $\sum |u_n|$ is convergent.

Theorem 53. If $\sum u_n$ is absolutely convergent, then $\sum u_n$ is convergent.

Theorem 54. Given a series $\sum u_n$, if a_n is the subsequence of u_n corresponding to the positive terms.If b_n is the subsequence of u_n corresponding to the non positive terms (possibly completed by 0 terms is one of the sequence is finite)

- If $\sum u_n$ is absolutely convergent, then $\sum a_n$ and $\sum b_n$ are convergent and $\sum u_n = \sum a_n + \sum b_n$.
- If $\sum u_n$ is conditionally convergent, then both $\sum a_n$ and $\sum b_n$ are divergent.

Definition 55. If $\sum u_n$ is convergent and not absolutely convergent, then $\sum u_n$ is conditionally convergent.

Theorem 56. (Alternating series) A series $\sum u_n$ such that its terms alternate between positive and negative, and such that $|u_n|$ is decreasing toward 0 is convergent.

8 Chapter 20

8.1 Pointwise convergence

Definition 57. Given a sequence of functions f_n ,
We say that f_n is convergent to f on an interval I if for any x in I, $\lim_{n\to\infty} f(x) = f(x)$.
Equivalently

$$\forall \epsilon > 0, \forall x \in I, \exists N_{\epsilon,x} > 0 \text{ such that } \forall n > N, |f_n(x) - f(x)| < \epsilon$$

Examples:

- $f_n(x) = \sin\left(\frac{x}{n}\right), \qquad x \in [0, \pi].$
- $f_n(x) = x^n, \qquad x \in [0, 1].$
- $f_n(x) = \begin{cases} 0, & x \in [0, n] \\ x - n, & x \in (n, n+1] \\ 1 & x \in (n+1, \infty) \end{cases}$.

8.2 Uniform convergence

Definition 58. A sequence f_n converges uniformly to f on an interval I if

$$\forall \epsilon > 0, \exists N_\epsilon > 0 \text{ such that } \forall n > N \text{ and } \forall x \in I, |f_n(x) - f(x)| < \epsilon$$

Theorem 59. (Cauchy uniform criterion) A sequence of functions f_n is uniformly convergent if

$$\forall \epsilon > 0, \exists N > 0 \text{ such that } \forall n > N, \forall p > N, |f_n(x) - f_p(x)| < \epsilon, \forall x \in I$$

Theorem 60. (Uniform convergence for series of functions) Let $\sum u_n(x)$ be a series of functions defined on an interval I. Let M_n a sequence of upper bounds for $|u_n(x)|$ on I ($M_n \geqslant |u_n(x)|$ for any x in I) , if $\sum M_n$ is convergent,
Then $\sum u_n(x)$ converges uniformly on I.

8.3 Result about uniform convergent sequences

Theorem 61. (Continuity of the limit) If a sequence of continuous functions f_n on an interval I is uniformly convergent to f on I, then f is continuous on I.

Theorem 62. (Integral of the limit) If f_n is a sequence of continuous functions on an interval I that is uniformly convergent to f on I, Then f is integrable on I and for any a and b in I,

$$\int_a^b f(t)\mathrm{d}t = \lim_{n\to\infty} \int_a^b f_n(t)\mathrm{d}t$$

Theorem 63. (Differentiability of the limit) If f_n is a sequence of differentiable function on an interval I such that

- $f_n(x)$ is convergent to $f(x)$ for any x in I
- f'_n converges uniformly to a function g on I.
- f'_n is continuous on I

Then f is differentiable, its derivative is g and $f' = g$ is continuous.