Differential Equations
Math 308, Fall 2012

Texas A&M University

Last Name (PRINT): Key
First Name (PRINT): ____________
ID No.: ____________

Fall 2012
Differential Equations
Exam 1

Instructions

1. The use of all electronic devices and documents is prohibited.
2. Present your solutions in the space provided. Show all your work neatly and concisely. Clearly indicate your final answer. You will be graded not merely on the final answer, but also on the quality and correctness of the work leading up to it.

The Aggie code of honor

"An Aggie does not lie, cheat, or steal or tolerate those who do." Scholastic dishonesty will not be tolerated.

The work on this test is my own.

Signature:____________

Questions	1	2	3	4	5	6	7	8
Grade:								

1. (10points) Given the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4x}{y}.$$

(a) Describe the differential equation (ordinary differential equation, partial differential equation, order, linear/non linear, autonomous/non autonomous, separable/non separable...)

ordinary order 1 non linear, non autonomous
separable.

(b) Prove that $y = 2x$ and $y = -2x$ are solutions, provided $x \neq 0$.

$y = 2x$ $\quad y'(x) = 2$

$$2 = \frac{4x}{2x} \Rightarrow \text{solution}$$

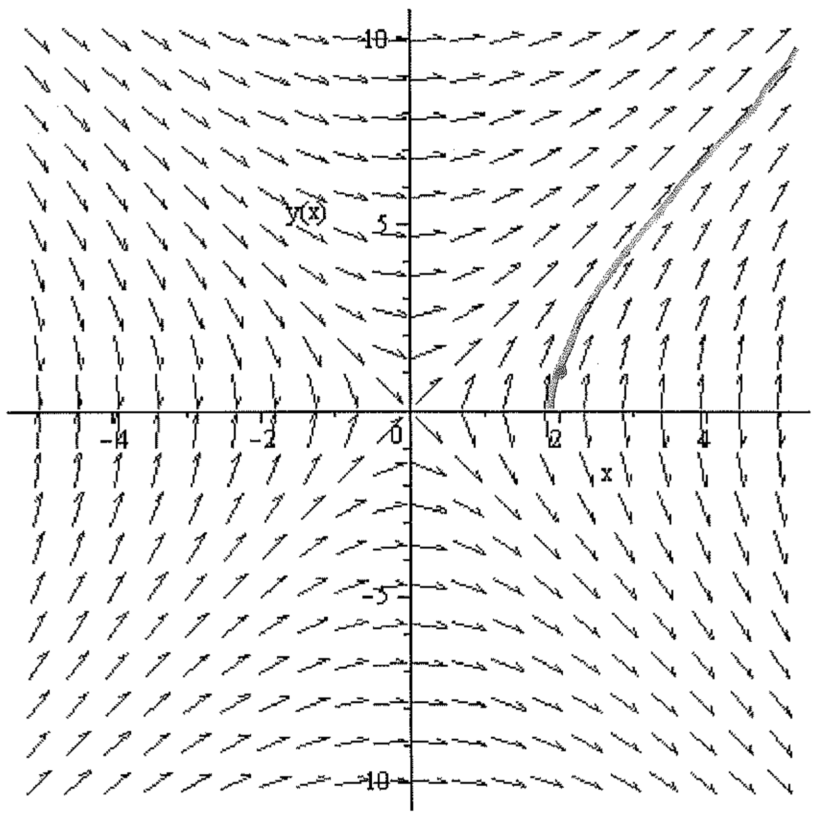
$y = -2x$ $\quad y'(x) = -2$

$$\frac{4x}{-2x} = -2 = y'(x) \Rightarrow \text{solution}.$$

(c)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4x}{y}.$$

The direction field is shown below



Sketch the solution curve with the initial condition $y(2) = 1$.
What is the domain of the solution you traced? (an approximative answer coherent with your graph is expected).

Domain $(+1.7, \infty)$

2. (8 points) Given the differential equation

$$\frac{dy}{dx} + 2xy = 3x^2\, e^{-x^2}.$$

(a) Find the value m such that for any real a, the function $y_a(x) = (x^m + a)\, e^{-x^2}$ is solution to the differential equation.

$$y_a'(x) = \left(m x^{m-1} - 2x\,(x^m + a)\right) e^{-x^2}$$

$$y_a'(x) + 2xy = \left[\left(m x^{m-1} - 2x^{m+1} - 2ax\right) + \left(2x^{m+1} + 2ax\right)\right] e^{-x^2}$$

$$= m x^{m-1} e^{-x^2} = 3x^2 e^{-x^2}$$

$$\Rightarrow \boxed{m = 3}$$

(b) Solve the initial value problem

$$\frac{dy}{dx} + 2xy = 3x^2\, e^{-x^2}, \quad y(0) = 6.$$

$$y = (x^3 + a)\, e^{-x^2}$$

$$y(0) = 6 = a \Rightarrow a = 6$$

$$\boxed{y = (x^3 + 6)\, e^{-x^2}}$$

3. (12 points) Solve the initial value problems

(a)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 8x^3\,\mathrm{e}^{-2y}, \quad y(1) = 0$$

separable $e^{2y}\,dy = 8x^3\,dx$

$$\frac{e^{2y}}{2} = \frac{8x^4}{4} + C \qquad e^{2y} = 4x^4 + C$$

$x=1,\ y=0 \quad 1 = 4 + C \qquad C = -3$

$$2y = \ln(4x^4 - 3)$$

$$\boxed{y = \frac{1}{2}\ln(4x^4 - 3)}$$

(b)

$$t\frac{\mathrm{d}y}{\mathrm{d}t} + 3y = t \quad y(2) = 3.$$

$$y' + \frac{3}{t} = 1$$

Integrating Factor $\mu = \exp\left(\int \frac{3}{t}\right) = \exp(3\ln(t)) = t^3$

$$t^3y' + 3t^2y = t^3$$

$$t^3y = \frac{t^4}{4} + C$$

$$y = \frac{t}{4} + \frac{C}{t^3}$$

$$y(2) = 3 = \frac{2}{4} + \frac{C}{8}$$

$C = 20 \Rightarrow$

$$\boxed{y = \frac{20}{t^3} + \frac{t}{4}}$$

4. (16points) A 100gal tank initially contains 20 gal of pure water. A salt-water solution containing 0.5lb of salt per gallon of water begins entering the tank at a rate of 4 gal/minute. Simultaneously, a drain is opened at the bottom of the tank that allows the salt-water mixture to leave the tank at a rate of 2 gallons per minute.

   (a) Find the volume $V(t)$ (in gal) of the salt-water mixture at any time t (in minutes) before the tank is full.
   At what time will the tank start to overflow?

$V(t) = 20 + 2t$

overflow when $V(t) = 100 = 20 + 2t$

$t = 40$ min

   (b) Let $A(t)$ be the amount of salt (in lb) in the tank.
   Find a differential equation satisfied by $A(t)$ before the tank is filled.
   State an initial condition.

$$\frac{dA(t)}{dt} = \text{rate in} - \text{rate out}$$
$$4 \times 0.5 - 2 \times \frac{A(t)}{V(t)}$$

$$\frac{dA(t)}{dt} = 2 - \frac{2A}{20+2t} = 2 - \frac{A}{10+t}$$

At $t = 0$ $A(t) = 0$

(c) What is the amount of salt in the tank at the precise moment when the tank gets full.

$$A'(t) + \frac{A}{10+t} = 2$$

integrating factor $\mu = \exp\left(\int \frac{1}{10+t}\,dt\right) = \exp(\ln(10+t))$

$$= (10+t)$$

$$(10+t)A'(t) + A = 20 + 2t$$

$$(10+t)A(t) = 20t + t^2 + C$$

$A(0) = 0 \Rightarrow C = 0$

$$A = \frac{20t + t^2}{10+t}$$

$A(40) = 48$ lb

5. (14points) Given the initial value problem

$$(*) \quad (t^2-1)y' + 2ty = t^2 + t, \quad y(t_0) = 0.$$

(a) State the **most** appropriate theorem of existence and uniqueness for the differential equation above.

given a linear differential equation of order 1 in the form $y' + p(x)\,y = q(x)$

If p, q are continuous on an interval I, for any x_0 in I and any y_0 there exists a unique solution

(b) For which values of t_0, the existence and uniqueness theorem **does not** apply?

to the initial value problem $y(x_0) = y_0$

b) $(*)$ equivalent to $y' + \frac{2t}{t^2-1}\,y = \frac{t^2+t}{t^2-1}$

the theorem does not apply for $t = 1$ and $t = -1$

(c) When the theorem applies, give the interval (possibly depending on t_0) on which the solution to the initial value problem $y(t_0) = 0$ exists and is unique.

IF t_0 in $(-\infty, -1)$ the solution to the initial value problem exists and is unique on $(-\infty, -1)$

IF t_0 in $(-1, 1)$ ——— $(-1, 1)$

IF t_0 in $(1, \infty)$ ——— on $(1, \infty)$

(d) Solve the initial value problem $y(0) = 3$.

Integrating factor $\mu = \exp\left(\int \frac{2t}{t^2-1}\right) = \exp\left(\ln(t^2-1)\right) = (t^2-1)$

$$(t^2-1)y = \frac{t^3}{3} + \frac{t^2}{2} + C$$

$y(0) = 3$

$-y(0) = C \quad C = -3$

$$y = \frac{2t^3 + 3t^2 - 18}{6(t^2-1)}$$

6. (18 points) Given the differential equation

$$y' = y^3 - 2y^2 + y$$

(a) Describe the differential equation (order, liner/non linear, autonomous/non autonomous, separable/non separable...)

order 1, non linear autonomous, separable

(b) Find the equilibrium solutions of the equation.

equilibrium solutions when $y' = 0$

$$0 = y^3 - 2y^2 - y = y(y-1)^2$$

2 equilibrium solutions $y = 0$, $y = 1$

(c) Sketch the phase line for the problem.
Classify each equilibrium point as asymptotically stable, semi-stable, or unstable.

0 unstable

1 semi-stable

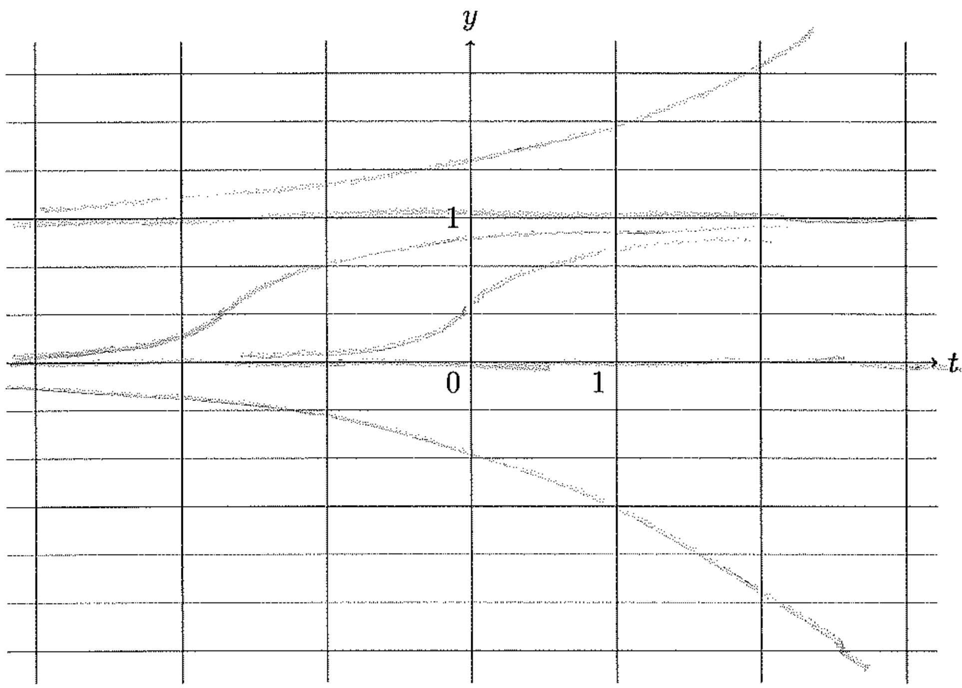
(d) For which values of y are the solutions concave up?

$$y''(x) = (3y^2 - 4y + 1)y'$$
$$= (3y-1)(y-1)y(y-1)^2$$
$$= (3y-1)(y-1)^3 y$$

y concave up if $y'' > 0$

$$\left(0, \tfrac{1}{3}\right) \cup (1, \infty)$$

(e) Sketch the graph of the equilibrium solutions and the graph of some solutions:



7. (10 points) Given the equation

$$\underbrace{(1 - y\sin x)}_{M} + \underbrace{(\cos x - 2)}_{N}\frac{dy}{dx} = 0.$$

(a) Determine whether the equation is exact.

$$\frac{\partial M}{\partial y} = -\sin x$$

$$\|$$

$$\frac{\partial N}{\partial x} = -\sin x$$

$\Rightarrow$ equation is exact

(b) Solve the initial value problem

$$(1 - y\sin x) + (\cos x - 2)\frac{dy}{dx} = 0, \quad y(0) = 5.$$

Give an explicit solution for y if possible.

$$\boxed{y = \frac{-5 - x}{\cos x - 2}}$$

$$\frac{\partial F}{\partial x} = 1 - y\sin x$$

$$\Rightarrow F = x + y\cos x + \beta(y)$$

$$\frac{\partial F}{\partial y} = \cos x + \beta'(y) = \cos x - 2$$

$$\beta'(y) = -2 \qquad \beta(y) = -2y + C$$

$$F = x + y\cos x - 2y + C = 0$$

$$y(0) = 5 \qquad 5 - 10 + C = 0 \qquad C = 5$$

8. (15 points) Given the differential equation

$$y'(x) = (6x+6)y^{2/3}(x).$$

(a) Find the general solution to the equation.

$$\frac{y'}{y^{2/3}} = 6x+6$$

$$3y^{1/3} = 3x^2 + 6x + C$$

$$y = \left(x^2 + 2x + C\right)^3$$

(b) Show the $y(x) = 0$ and $y(x) = x^3(x+2)^3$ at two solutions to the initial value problem

$$y' = (6x+6)y^{2/3}, \quad y(0) = 0.$$

IF $y = 0$, $y' = 0$

$(6x+6)y^{2/3} = 0 = y'$

and $y(0) = 0$

IF $y(x) = x^3(x+2)^3$ is solution from (a) $(c = 0)$

and $y(0) = 0$.

(c) Does it contradict the theorem of existence and uniqueness? Why?

No because the equation

$y' = f(x,y)$ does not satisfy the condition

$\frac{\partial f}{\partial y}$ continuous at 0.