Differential Equations
Math 308, Fall 2012

Texas A&M University

Last Name (PRINT): Key
First Name (PRINT): ____________
ID No.: ____________

solution

Fall 2012
Differential Equations
Exam 2

Instructions

1. The use of all electronic devices and documents is prohibited.
2. Present your solutions in the space provided. Show all your work neatly and concisely. Clearly indicate your final answer. You will be graded not merely on the final answer, but also on the quality and correctness of the work leading up to it.

The Aggie code of honor

"An Aggie does not lie, cheat, or steal or tolerate those who do." Scholastic dishonesty will not be tolerated.

The work on this test is my own.

Signature: ____________

Questions	1	2	3	4	5	6	7	8
Grade:								

1. (12 points) Solve the initial value problem

$$y'' + 2y' + 2y = 0, \qquad y(0) = 2, \quad y'(0) = -3.$$

$r^2 + 2r + 2 = 0 \qquad r = -1 + i$

$(r+1)^2 + 1 = 0 \qquad r = -1 - i$

general solution $\quad y = c_1 e^{-x} \cos x + c_2 e^{-x} \sin x$

$y(0) = 2 \Rightarrow c_1 = 2$

$y'(x) = -2e^{-x} \cos x - 2e^{-x} \sin x + c_2 e^{-x} \cos x - c_2 e^{-x} \sin x$

$y'(0) = -3 = -2 + c_2 \qquad c_2 = -1$

solution $\quad y = 2e^{-x} \cos x - e^{-x} \sin x$

2. (13points) The function $y_1(x) = x^2$ is solution to the differential equation

$$x^2y'' - 2y = 0, \qquad x > 0.$$

Find a fundamental set of solutions (find a second solution and check that the 2 functions form a fundamental set of solutions).

second solution in the form

$$y = \lambda(x)\, x^2$$

$$y' = 2x\, \lambda(x) + \lambda'(x)\, x^2$$

$$y'' = 2\lambda(x) + 4x\,\lambda'(x) + \lambda''(x)\, x^2$$

$$0 = x^2 y'' - 2y = 2x^2\lambda(x) + 4x^3\lambda'(x) + x^4\lambda''(x) - 2\lambda(x)x^2$$

$$4x^3\lambda'(x) + x^4\lambda''(x) = 0$$

$$\frac{\lambda''(x)}{\lambda'(x)} = -\frac{4x^3}{x^4} = \frac{-4}{x}$$

$$\ln(\lambda') = -4\ln(x) + C = \ln(x^{-4}) + C$$

$$\lambda' = \frac{C}{x^{-4}}$$

$$\lambda = \frac{C_1}{x^{-3}} + C_2$$

second solution $y_2 = \dfrac{x^2}{x^{-3}} = \dfrac{1}{x}$

$$W(y_1, y_2) = \begin{pmatrix} x^2 & \frac{1}{x} \\ 2x & \frac{-1}{x^2} \end{pmatrix} = -3 \neq 0$$

linearly independent

3. (10 points)

   (a) If you were using the method of undetermined coefficients, what is your best guess for the form of a particular solution to

$$y'' - 2y' + 3y = 3\sin x.$$

(You do not have to solve this)

solution to the homogeneous equation

$r^2 - 2r + 3 = 0$

$r = 3$ or $r = -1$

$y = c_1 e^{3x} + c_2 e^{-x}$

Guess $y = a\sin x + b\cos x$

   (b) If you were using the method of undetermined coefficients, what is your best guess for the form of a particular solution of the differential equation

$$y'' - 2y' + y = e^x$$

(You do not have to solve this)

solution to the homogeneous equation

$r^2 - 2r + 1 = 0$ $\quad y = c_1 e^x + c_2 x e^x$

$r = 1$ repeated

Guess $ax^2 e^x$

4. (15 points) Find the solution of the initial value problem

$$y'' - 2y' = 4x, \qquad y(0) = 0, \quad y'(0) = 0$$

characteristic equation

$r^2 - 2r = 0$

$r = 0$ or $r = 2$

general sol to the homogeneous problem $y = C_1 + C_2 e^{2x}$

Particular solution:

Guess $x^1(ax + b) = ax^2 + bx$

$y'(x) = 2ax + b$

$y''(x) = 2a$

$$\boxed{y = -x(x+1) + \frac{1}{2}\left(e^{2x} - 1\right)}$$

$2a - 2(2ax + b) = 4x$

$-4ax + 2a - 2b = 4x$

$$\begin{cases} -4a = 4 \\ 2a - 2b = 0 \end{cases}$$

$a = -1$

$b = -1$

solution $y = -x^2 - x + C_1 + C_2 e^{2x}$

$$\boxed{C_1 = -\frac{1}{2}}$$

$y(0) = 0 \Rightarrow C_1 + C_2 = 0$

$y'(x) = -2x - 1 + 2C_2 \quad y'(0) = 0 \quad \boxed{C_2 = \frac{1}{2}}$

5. (15points) Use the method of variation of parameters to find the general solution to

$$y'' - 2y' + y = \frac{e^x}{x}$$

* solution to the homogeneous problem

$r^2 - 2r + 1 = 0$

$(r-1)^2 = 0$

$c_1 e^x + c_2 x e^x$

* solution to the non homogeneous problem in the form

$c_1(x) e^x + c_2(x)\ x e^x$

$$\begin{cases} c_1' e^x + c_2' x e^x = 0 \\ c_1' e^x + c_2' (x e^x + e^x) = \dfrac{e^x}{x} \end{cases}$$

$= c_2' e^x = \dfrac{e^x}{x}$

$c_2' = \dfrac{1}{x}$

$c_2 = \ln x$

$c_1' + x c_2' = 0 \Rightarrow c_1' = -1 \quad c_1 = -x$

<u>general solution</u>

$\boxed{-x e^x + x \ln x\, e^x + c_1 e^x + c_2 x e^x}$

or $\boxed{C_2 x e^x + C_1 e^x + x \ln x\, e^x}$

6. (8 point) Find the Laplace transform of the function

$$f(x) = \begin{cases} 5 & 0 \leqslant x \leqslant 1 \\ e^{2x} & 1 < x. \end{cases}$$

By definition the laplace transform of F

is $\int_0^\infty e^{-st} f(t)\, dt$

$$\mathcal{L}\{f(x)\}(s) = \int_0^1 5e^{-st}\, dt + \int_1^\infty e^{2t} e^{-st}\, dt$$

$$= \frac{5}{-s} e^{-st}\Big|_0^1 + \frac{e^{(2-s)t}}{2-s}\Big|_1^\infty$$

$$\boxed{= \frac{5}{s} - \frac{5}{s}e^{-s} + \frac{e^{2-s}}{s-2}}$$

7. (10 points) Find the inverse Laplace transform of

$$F(s) = \frac{s+5}{(s-1)(s+2)}.$$

$$F(s) = \frac{a}{s-1} + \frac{b}{s+2} = \frac{2}{s-1} - \frac{1}{s+2}$$

$$\mathcal{L}^{-1}\{F(s)\}(t) = 2e^{t} - e^{-2t}$$

8. (17 points) Given the initial value problem

$$y'' - y = 2 \qquad y(0) = 0, \quad y'(0) = 1.$$

(a) Let y be the solution of the initial value problem and $\mathcal{L}\{y\}(s)$ be the Laplace transform of y.
Find $\mathcal{L}\{y\}(s)$.

$$s^2 \mathcal{L}\{y\} - 1 - \mathcal{L}\{y\} = \mathcal{L}\{2\} = \frac{2}{s}$$

$$(s^2 - 1)\mathcal{L}\{y\} = \frac{2}{s} + 1$$

$$\boxed{\mathcal{L}\{y\} = \frac{2+s}{s(s^2-1)}}$$

(b) Find the solution y of the initial value problem using Laplace transforms.
(No partial credit will be given for finding the solution using a method that does not involve Laplce transforms. You may use the result from part (a) if needed).

$$\mathcal{L}\{y\}(s) = \frac{a}{s} + \frac{b}{s-1} + \frac{c}{s+1}$$

$$= \frac{-2}{s} + \frac{1}{2(s+1)} + \frac{3}{2(s-1)}$$

$$\boxed{y = -2 + \frac{1}{2}e^{-t} + \frac{3}{2}e^{t}}$$