Differential Equations
Math 308, Fall 2012

Texas A&M University

Last Name (PRINT): Key
First Name (PRINT):
ID No.:

Solutions

Fall 2012
Differential Equations
Exam 3

Instructions

1. The use of all electronic devices and documents is prohibited.
2. Present your solutions in the space provided. Show all your work neatly and concisely. Clearly indicate your final answer. You will be graded not merely on the final answer, but also on the quality and correctness of the work leading up to it.

The Aggie code of honor

"An Aggie does not lie, cheat, or steal or tolerate those who do." Scholastic dishonesty will not be tolerated.

The work on this test is my own.

Signature:

Questions	1	2	3	4	5	6	7
Grade:							

1. (15 points) Find the inverse Laplace transform of the function

$$F(s) = \frac{2 - e^{-2s}}{s^2 + 2s + 2} + \frac{1 - e^{-s}}{s^2}$$

$$F(s) = \frac{2}{(s+1)^2+1} - \frac{e^{-2s}}{(s+1)^2+1} + \frac{1}{s^2} - \frac{e^{-s}}{s^2}$$

$$f(t) = 2e^{-t}\sin t - u_2(t)\, e^{-(t-2)}\sin(t-2) + t - u_1(t)(t-1)$$

2. (15points)Consider the initial value problem

$$y'' + y = g(t) \qquad y(0) = 0 \quad y'(0) = 0, \quad \text{where } g(t) = \begin{cases} 3t & \text{for } 0 \leqslant t < 1 \\ 3 & \text{for } 1 \leqslant t \end{cases}$$

(a) Find the Laplace transform of the function g.

$$g(t) = 3t + (3 - 3t)u_1(t)$$
$$= 3t - 3(t-1)u_1(t)$$

$$\mathcal{L}\{g\}(s) = \frac{3}{s^2} - 3\frac{e^{-s}}{s^2}$$

(b) Find the solution to the initial value problem.

$$Y(s) = \mathcal{L}\{y\}(s)$$

$$s^2 Y(s) - 0 - 0 + Y(s) = \frac{3 - 3e^{-s}}{s^2}$$

$$Y(s) = \frac{3(1 - e^{-s})}{s^2(s^2+1)} \qquad \frac{3}{s^2(s^2+1)} = \frac{3}{s^2} - \frac{3}{s^2+1}$$

$$Y(s) = \frac{3}{s^2} - \frac{3}{s^2+1} - \frac{3e^{-s}}{s^2} + \frac{3e^{-s}}{s^2+1}$$

$$\boxed{y(t) = 3t - 3\sin t - 3(t-1)u_1(t) + 3\sin(t-1)u_1(t)}$$

3. (15 points) Find the solution to the initial value problem

$$y'' - 4y' - 5y = 3\delta(t-1), \qquad y(0) = 0, \quad y'(0) = 3.$$

$$s^2 Y(s) - 3 - 4sY(s) - 0 - 5Y(s) = 3e^{-s}$$

$$(s^2 - 4s - 5)Y(s) = 3e^{-s} + 3$$

$$Y(s) = \frac{3e^{-s} + 3}{s^2 - 4s - 5}$$

$$\frac{3}{s^2 - 4s - 5} = \frac{\frac{1}{2}}{s-5} - \frac{\frac{1}{2}}{s+1}$$

$$Y(s) = \frac{1}{2}\left(\frac{1}{s-5}\right) - \frac{1}{2}\left(\frac{1}{s+1}\right) + \frac{1}{2}\frac{e^{-s}}{s-5} - \frac{1}{2}\frac{e^{-s}}{s+1}$$

$$\boxed{y(t) = \frac{1}{2}e^{5t} - \frac{1}{2}e^{-t} + \frac{1}{2}e^{5(t-1)}u_1(t) - \frac{1}{2}e^{-(t-1)}u_1(t)}$$

4. (10 points) Write the initial value problem

$$y'' + (1-t^2)y' - 2ty = \cos t, \qquad y(0) = 1, \quad y'(0) = -3.$$

as a system of first order differential equations using the following matrix notation:

$$X' = AX + G, \quad X(0) = \begin{pmatrix} a \\ b \end{pmatrix}$$

(Determine $A(t)$, $G(t)$ and $X(0)$).

$X_1(t) = y(t)$

$X_2(t) = y'(t)$

$X_1'(t) = y'(t) = X_2(t)$

$X_2'(t) = y''(t) = 2t\, X_1(t) - (1-t^2)X_2(t) + \cos t$

$$X(t) = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \qquad X'(t) = \underbrace{\begin{pmatrix} 0 & 1 \\ 2t & t^2-1 \end{pmatrix}}_{A(t)} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ \cos t \end{pmatrix}}_{G(t)}$$

$$X(0) = \begin{pmatrix} y(0) \\ y'(0) \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ -3 \end{pmatrix}}_{X(0)}$$

5. (18 points) Consider the system

$$X' = AX \quad \text{with } A = \begin{pmatrix} -3 & -4 \\ 2 & 3 \end{pmatrix}$$

(a) Find the general solution to the system.

$$\det\begin{pmatrix} -3-\lambda & -4 \\ 2 & 3-\lambda \end{pmatrix} = (-3-\lambda)(3-\lambda) + 8$$

$$= \lambda^2 - 1 = (\lambda-1)(\lambda+1)$$

2 eigenvalues: $\lambda = 1$, $\lambda = -1$

eigenvector for 1 $\begin{pmatrix} -4 & -4 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{cases} -4x - 4y = 0 \\ 2x + 2y = 0 \end{cases} \quad \boxed{x = -y} \qquad V_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

eigenvector for -1

$$\begin{pmatrix} -2 & -4 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \begin{matrix} -2x - 4y = 0 \\ 2x + 4y = 0 \end{matrix} \quad x = -2y \quad V_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

general solution:

$$X(t) = c_1 e^{-t}\begin{pmatrix} -2 \\ 1 \end{pmatrix} + c_2 e^{t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

(b) Find the solution to the initial value problem

$$X' = AX, \qquad X(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

$$X(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2c_1 \\ c_1 \end{pmatrix} + \begin{pmatrix} c_2 \\ -c_2 \end{pmatrix}$$

$$1 = -2c_1 + c_2 \qquad c_1 = -1$$

$$0 = c_1 - c_2 \qquad c_1 = c_2$$

$$X(t) = -e^t \begin{pmatrix} 1 \\ -1 \end{pmatrix} - e^{-t} \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

(c) Describe the behavior as $t \to \infty$ of the solution to the initial value problem.

$t \to \infty$

The direction $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is asymptote to $X(t)$ at ∞

6. (15 points) Given a system of differential equations in the form

$$X'(t) = A(t)X(t) + G(t) \quad t > 0$$

where $A(t)$ is a continuous 3×3 matrix and $G(t)$ is a continuous vector function for $t > 0$. The vectors

$$X_1(t) = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \quad X_2(t) = \begin{pmatrix} 2t \\ 0 \\ 3t \end{pmatrix} \quad X_3(t) = \begin{pmatrix} 3t^2 \\ 2t^2 \\ t^2 \end{pmatrix}$$

are solutions to the corresponding homogeneous equation.

The vector

$$X_p(t) = \begin{pmatrix} \frac{1}{t} \\ \frac{-2}{t} \\ \frac{2}{t} \end{pmatrix}$$

is a particular solution to the non-homogeneous problem.

(a) Determine whether the vectors $X_1(t)$, $X_2(t)$, $X_3(t)$ are linearly independent for $t > 0$.

$$\det \begin{pmatrix} -1 & 2t & 3t^2 \\ 0 & 0 & 2t^2 \\ 0 & 3t & t^2 \end{pmatrix} = (-1) \begin{vmatrix} 0 & 2t^2 \\ 3t & t^2 \end{vmatrix} = (-1)(-6t^3) \neq 0$$

when $t > 0$

$\Rightarrow X_1(t), X_2(t)\ X_3(t)$ are linearly independent.

(b) Find a solution to the non-homogeneous problem that satisfies the condition

$$X(1) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

$$X(t) = X_p + C_1 X_1(t) + C_2 X_2(t) + C_3 X_3(t)$$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} + \begin{pmatrix} -C_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 2C_2 \\ 0 \\ 3C_2 \end{pmatrix} + \begin{pmatrix} 3C_3 \\ 2C_3 \\ C_3 \end{pmatrix}$$

$C_3 = 1$

$2 + 3C_2 + 1 = 0 \qquad C_2 = -1$

$1 - C_1 + 2C_2 + 3C_3 = 0$

$C_1 = 1 - 2 + 3 = 2$

$$\boxed{X(t) = \begin{pmatrix} \frac{1}{t} \\ \frac{-2}{t} \\ \frac{2}{t} \end{pmatrix} + 2\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 2t \\ 0 \\ 3t \end{pmatrix} + \begin{pmatrix} 3t^2 \\ 2t^2 \\ t^2 \end{pmatrix}}$$

7. (12 points) Given the system

$$\begin{cases} x_1' = -2x_1 - 5x_2 \\ x_2' = x_1 + 2x_2 \end{cases} \qquad x_1(0) = 1, x_2(0) = 0$$

(a) Transform the system into a second order initial value problem.

$x_1 = x_2' - 2x_2$

$x_1' = x_2'' - 2x_2' = -2(x_2' - 2x_2) - 5x_2$

$x_2'' - 2x_2' = -2x_2' + 4x_2 - 5x_2$

$\boxed{x_2'' + x_2 = 0}$

initial conditions: $x_2(0) = 0$

$x_2'(0) = x_1(0) + 2x_2(0) = 1$

(b) Using the previous question, find the solutions $x_1(t)$, $x_2(t)$.

$x_2'' + x_2 = 0 \qquad x_2(0) = 0 \qquad x_2'(0) = 1$

$x_2 = a\cos t + b\sin t$

$x_2(0) = 0 \Rightarrow a = 0$

$x_2'(0) = 1 = b$

$\Rightarrow \boxed{\begin{aligned} x_2 &= \sin t \\ x_1 &= x_2' - 2x_2 = \cos t - 2\sin t \end{aligned}}$