University of Washington

Review for Final Exam

1 Chapter 7

Exercise 1. Transform the initial value problem into an initial value problem for a system of first order differential equation

1. (5p364) $u'' + 0.25u' + 4u = 2\cos 3t, \qquad u(0) = 1, \quad u'(0) = -2.$
2. $\begin{cases} y'' + x' - 5y = \sin t \\ x'' - 3y' + 2x' - 7y = \cos t \end{cases} \qquad x(0) = 1, \quad x'(0) = -3, \quad y(0) = 2, \quad y'(0) = -5.$

Exercise 2. (10p364) Transform the given system into a single equation of second order. Find x_1 and x_2 that satisfy the initial condition

$$\begin{cases} x_1' = x_1 - 2x_2, & x_1(0) = -1 \\ x_2' = 3x_1 - 4x_2, & x_2(0) = 2 \end{cases}$$

Exercise 3. Given the matrices

$$A = \begin{pmatrix} -2 & 1 & 2 \\ 1 & 0 & -3 \\ 2 & -1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 & 3 \\ 3 & -1 & -1 \\ -2 & 1 & 0 \end{pmatrix}$$

1. Find $3A - 2B$
2. Calculate $\det(A)$.
3. Find $A^2 - AB$.

Exercise 4. Verify that the given vector/matrix satisfies the given differential equation

1. (24p378) $X'(t) = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} X(t), \qquad X(t) = \begin{pmatrix} 6 \\ -8 \\ -4 \end{pmatrix} e^{-t} + 2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{2t}.$
2. (26p378) $X'(t) = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} X(t), \qquad X(t) = \begin{pmatrix} e^t & e^{-2t} & e^{3t} \\ -4e^t & -e^{-2t} & 2e^{3t} \\ -e^t & -e^{-2t} & e^{3t} \end{pmatrix}.$

Exercise 5. Solve the given system of equations, or else show that there is no solution.

1. (2p388) $\begin{cases} x_1 + 2x_2 - x_3 = 1 \\ 2x_1 + x_2 + x_3 = 1 \\ x_1 - x_2 + 2x_3 = 1 \end{cases}$
2. (1p388) $\begin{cases} x_1 - x_3 = 0 \\ 3x_1 + x_2 + x_3 = 0 \\ -x_1 + x_2 + 2x_3 = 0 \end{cases}$

3. (6p388) $\begin{cases} x_! + 2x_2 - x_3 = -2 \\ -2x_1 - 4x_2 + 2x_3 = 4 \\ 2x_1 + 4x_2 - 2x_3 = -4 \end{cases}$

Exercise 6. (ex 2 WIR 12) Are the vector functions linearly independent? If they are linearly dependent, find a linear relation among them.

1. $X_1(t) = \begin{pmatrix} e^{-3t} \\ -4e^{-3t} \end{pmatrix} \qquad X_2(t) = \begin{pmatrix} e^{-3t} \\ e^{-3t} \end{pmatrix}$.
2. $X_1(t) = \begin{pmatrix} 2e^t \\ -e^t \end{pmatrix}, \qquad X_2(t) = \begin{pmatrix} e^t \\ 2e^t \end{pmatrix} \qquad X_3(t) = \begin{pmatrix} 3te^t \\ te^t \end{pmatrix}$.
3. $X_1(t) = \begin{pmatrix} e^{3t} \\ e^{3t} \end{pmatrix} \qquad X_2(t) = \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}$.
4. $X_1(t) = \begin{pmatrix} 2\cos t \\ 3\sin t \\ \tan t \end{pmatrix}, \qquad X_2 = \begin{pmatrix} \cos t \\ 2\sin t \\ \tan t \end{pmatrix}, \qquad X_3 = \begin{pmatrix} \cos t \\ 0 \\ -\tan t \end{pmatrix}$.

Exercise 7. Find the eigenvalues and eigenvectors of the given matrix

1. (18p389) $A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$.
2. (17p389) $A = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}$.
3. (23p389) $A = \begin{pmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ -2 & -4 & -1 \end{pmatrix}$.
4. (25p389) $A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$.

Exercise 8. Consider the system

$$X'(t) = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} X(t)$$

1. Show that the vectors $X_1(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}, \qquad X_2(t) = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$ are solutions.
2. Are they linearly independent? Describe all the solutions to the system.
3. Find the solution to the initial value problem $X(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

Exercise 9. Find the general solution of the given equations and describe the behavior of the solution as $t \to \infty$.

1. (1p405) $X' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} X$.
2. (2p405) $X' = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} X$.

3. (13p405) $X' = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -8 & -5 & -3 \end{pmatrix} X.$

4. (1p417) $X' = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} X.$

5. (6p409) $X' = \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix} X.$

6. (2p417) $X' = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} X.$

7. (8p417) $X' = \begin{pmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{pmatrix} X.$

8. (10p436) $X' = \begin{pmatrix} 3 & 9 \\ -1 & -3 \end{pmatrix} X \quad X(0) = \begin{pmatrix} 2 \\ 4 \end{pmatrix}.$

9. (5p436) $X" = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} X.$

10. (11p436) $X' = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 3 & 6 & 2 \end{pmatrix} X, \qquad X(0) = \begin{pmatrix} -1 \\ 2 \\ -30 \end{pmatrix}$

Exercise 10. (18p405) Solve the initial value problem.

$$X' = \begin{pmatrix} 0 & 0 & -1 \\ 2 & 0 & 0 \\ -1 & 2 & 4 \end{pmatrix} X \qquad X(0) = \begin{pmatrix} 7 \\ 5 \\ 5 \end{pmatrix}.$$

Exercise 11. Find the general solution of the system

1. (7p447) $X' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} X + \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^t.$

2. (4p447) $X' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} X + \begin{pmatrix} \mathrm{e}^{-2t} \\ -2\,\mathrm{e}^t \end{pmatrix}.$

3. (2p447) $X' = \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} X + \begin{pmatrix} e^t \\ \sqrt{3}e^{-t} \end{pmatrix}.$

4. (5p447) $X' = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} X + \begin{pmatrix} t^{-2} \\ t^{-3} \end{pmatrix}$

5. (8p447) $X' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} X + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t.$

2 Section 10.1

Either solve the given boundary problem or else show that it has no solution.

Exercise 12. (1p595) $y'' + y = 0, \quad y(0) = 0, \quad y'(\pi) = 1.$

Exercise 13. (3p595) $y'' + y = 0, \quad y(0) = 0, \quad y(L) = 0.$

Exercise 14. (4p595) $y'' + y = 0, \quad y'(0) = 1, \quad y(L) = 0.$

Exercise 15. (5p595) $y'' + y = x, \quad y(0) = 0, \quad y(\pi) = 0.$

Exercise 16. (9p595) $y'' + 4y = \cos x, \quad y(0) = 0, \quad y'(\pi) = 0$

In each problem, find the eigenvalues and the eigenfunctions of the given boundary value problem. Assume that all eigenvalues are real.

Exercise 17. (14p595) $y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(\pi) = 0.$

Exercise 18. (16p595) $y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(\pi).$

3 Section 10.2

Exercise 19. (9p605) If $f(x) = -x$ for $-L < x \leqslant L$, and if $f(x + 2L) = f(x)$, find a formula for $f(x)$ in the interval $L < x < 2l$; in the interval $-3L < x < -2L$.

In each problem

1. Sketch the graph of the function for 3 periods.
2. Find the Fourier series for the function.

Exercise 20. (13p605) $f(x) = -x, \quad -L \leqslant x < L, \quad f(x + 2L) = f(x).$

Exercise 21. (14p605) $f(x) = \begin{cases} 1, & -L \leqslant x < 0 \\ 0, & 0 \leqslant x < L \end{cases} \quad f(x + 2L) = f(x).$

Exercise 22. (15p605) $f(x) = \begin{cases} x, & -\pi \leqslant x < 0 \\ 0, & 0 \leqslant x < \pi \end{cases} \quad f(x + 2\pi) = f(x).$

Exercise 23. (16p605) $f(x) = \begin{cases} x + 1, & -1 \leqslant x < 0 \\ 1 - x, & 0 \leqslant x < 1 \end{cases} \quad f(x + 2) = f(x).$

4 Section 10.3

Assume that the function is periodically extended outside the original interval.

1. Find the Fourier series for the extended function.
2. Sketch the graph of the function to which the series converges for 3 periods.

Exercise 24. (1p612) $f(x) = \begin{cases} -1, & -1 \leqslant x < 0 \\ 1, & 0 \leqslant x < 1 \end{cases}.$

Exercise 25. (4p612) $f(x) = 1 - x^2, \quad -1 \leqslant x < 1.$

Exercise 26. (5p612) $f(x) = \begin{cases} 0, & -\pi \leqslant x < -\frac{\pi}{2} \\ 1, & -\frac{\pi}{2} \leqslant x < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} \leqslant x < \pi \end{cases}$

5 Section 10.4

In each problem, a function f is given on an interval of length L. In each case, sketch the graph of th even and of the odd extension of f of period $2L$.

Exercise 27. (9p620) $f(x) = 2 - x, \qquad 0 < x < 2.$

Exercise 28. (11p620) $f(x) = \begin{cases} 0, & 0 \leqslant x < 1 \\ 1, & 1 \leqslant x < 2 \end{cases}$

In each problem, a function f is given, sketch the graph of the odd extension and of the even extension of f.

Exercise 29. (7p620)$f(x) = \begin{cases} x, & 0 \leqslant x < 2 \\ 1, & 2 \leqslant x < 3 \end{cases}.$

Exercise 30. (10p620) $f(x) = x - 3, \qquad 0 < x < 4.$

Exercise 31. (12p620) $f(x) = 4 - x^2, \qquad 0 < x < 1.$

In each problem, find the Fourier series, and sketch the graph to which the series is convergent over 3 periods.

Exercise 32. (15p620) $f(x) = \begin{cases} 1, & 0 \leqslant x < 1 \\ 0, & 1 \leqslant x < 2 \end{cases}$. Cosine series, period 4.

Exercise 33. (16p620) $f(x) = \begin{cases} x & 0 \leqslant x < 1 \\ 1, & 1 \leqslant x < 2 \end{cases}$. Sine series, period 4

Exercise 34. (21p620) $f(x) = L - x, \quad 0 \leqslant x \leqslant L$. cosine series, period $2L$.

Exercise 35. (22p620) $f(x) = L - x, \quad 0 \leqslant x \leqslant L$. sine series, period $2L$.

In each problem, a function f is given on an interval of length L. In each case,

1. sketch the graph of th even extension $g(x)$ and the odd extension $h(x)$ of f of period $2L$ over 3 periods.
2. Find the Fourier cosine series (Fourier series of g) and the Fourier sine series (Fourier series of h).

Exercise 36. (27p621) $f(x) = 3 - x \qquad 0 < x < 3.$

Exercise 37. (28p621) $f(x) = \begin{cases} x, & 0 \leqslant x < 1 \\ 0, & 1 \leqslant x < 2 \end{cases}$

6 Section 10.5

In each problem, determine whether the method of separation of variables can be used to replace the given partial differential equation by a pair of ordinary differential equations. If so, find the equations.

Exercise 38. (1p630) $x\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial u}{\partial t} = 0.$

Exercise 39. (5p630) $\dfrac{\partial^2 u}{\partial x^2} + (x+y)\dfrac{\partial^2 u}{\partial y^2} = 0.$

Exercise 40. (6 p630) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + xu = 0.$

Exercise 41. (7p250) Find the solution to the heat conduction problem

$$\begin{aligned} &100\tfrac{\partial^2 u}{\partial x^2} = \tfrac{\partial u}{\partial t}, \qquad 0 < x < 1, \quad t > 0; \\ &u(0,t) = 0, \quad u(1,t) = 0 \qquad t > 0 \\ &u(x,0) = \sin(2\pi x) - \sin(5\pi x), \quad 0 \leqslant x \leqslant 1 \end{aligned}$$

Consider the conduction of heat in a rod 40 cm in length whose ends are maintained at 0°C for all $t > 0$. Find an expression for the temperature $u(x, t)$ if the initial temperature distribution in the rod is the given function. Suppose that $\alpha^2 = 1$.

Exercise 42. (9p630) $u(x, 0) = 50, \qquad 0 < x < 40$

Exercise 43. (10p630) $u(x, 0) = \begin{cases} x, & 0 \leqslant x < 20 \\ 40 - x, & 20 \leqslant x \leqslant 40 \end{cases}$.

7 Section 10.6

Find the steady state solution of the heat conduction equation

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

that satisfies the given set of boundary conditions.

Exercise 44. (1p639) $u(0, t) = 10, \qquad u(50, t) = 40$

Exercise 45. (3p639) $\frac{\partial u}{\partial x}(0, t) = 0, \qquad u(L, t) = 0$

Exercise 46. (7p639) $\frac{\partial u}{\partial x}(0, t) - u(0, t) = 0, \qquad u(L, t) = T$

Exercise 47. (9p639) Let an aluminum rod of length 20 cm be initially at the uniform temperature of 25°C. Suppose that at time $t = 0$, the end $x = 0$ is cooled to 0°C while the end $x = 20$ is heated to 60°, and both are thereafter maintained at those temperatures.

Find the temperature distribution in the rod at time t.

Exercise 48. (10p640)

1. Let the end of a copper rod 100 cm long be maintained at 0°C. Suppose that the center of the bar is heated to 100^{circ}C by an external heat source and that this situation is maintained until a steady state results. Find the steady state temperature distribution.

2. At time $t = 0$ (after the steady state from part 1. has been reached) , let the heat source be removed. At the same instant let the end $x = 0$ be placed in thermal contact with a reservoir at 20°C, while the other end remains at 0°C. Find the temperature as a function of the position and the time.

Exercise 49. (13p640) Consider a bar of length 40 cm whose initial temperature is given by $u(x, 0) = \dfrac{x(60 - x)}{30}$. Suppose that $\alpha^2 = 0.25$ cm^2/s and that both ends are insulated.

1. Find the temperature $u(x, t)$.

2. Determine the steady state temperature in the bar.

Exercise 50. Consider a bar 30 cm long that is made of a material for which $\alpha^2 = 1$ and whose ends are insulated, Suppose that the initial temperature is zero except for the interval $5 < x < 10$, where the initial temperature is 25°C.

Find the temperature $u(x, t)$.

8 Section 10.7

Consider an elastic string of length L whose ends are held fixed. The string is set in motion with no initial velocity from an initial position $u(x,0) = f(x)$.

Find the displacement $u(x,t)$ for the given initial function f.

Exercise 51. (1p652)$f(x) = \begin{cases} \frac{2x}{L}, & 0 \leqslant x \leqslant \frac{L}{2} \\ \frac{2(L-x)}{L} & \frac{L}{2} < x < L \end{cases}$

Exercise 52. (3p652) $f(x) = \dfrac{8x(L-x)^2}{L^3}$

Exercise 53. (4p652) $f(x) = \begin{cases} 1, & \frac{L}{2} - 1 \leqslant x \leqslant \frac{L}{2} + 1 \\ \frac{2(L-x)}{L} & \text{otherwise} \end{cases}$

Consider an elastic string of length L whose ends are held fixed. The string is set in motion from its equilibrium position with an initial velocity $\dfrac{\partial u}{\partial t}(x,0) = g(x)$.

Find the displacement $u(x,t)$ for the given function g.

Exercise 54. (5p653)$g(x) = \begin{cases} \frac{2x}{L}, & 0 \leqslant x \leqslant \frac{L}{2} \\ \frac{2(L-x)}{L} & \frac{L}{2} < x < L \end{cases}$

Exercise 55. (7p652) $f(x) = \dfrac{8x(L-x)^2}{L^3}$

Exercise 56. (8p652) $f(x) = \begin{cases} 1, & \frac{L}{2} - 1 \leqslant x \leqslant \frac{L}{2} + 1 \\ \frac{2(L-x)}{L} & \text{otherwise} \end{cases}$

9 Section 10.8

Exercise 57. (1p665)

1. Find the solution $u(x,y)$ of the Laplace's equation in the rectangle $0 < x < a$, $0 < x < b$, that satisfies the boundary conditions
$$u(0,y) = 0, \qquad u(a,y) = 0, \qquad 0 < y < b$$
$$u(x,0) = 0, \qquad u(x,b) = g(x), \qquad 0 \leqslant x \leqslant a$$

2. Find the solution if $g(x) = \begin{cases} x, & 0 \leqslant x \leqslant a/2, \\ a - x, & a/2 < x \leqslant a \end{cases}$

Exercise 58. (2p655-656) Find the solution $u(x,y)$ of the Laplace's equation in the rectangle $0 < x < a$, $0 < x < b$, that satisfies the boundary conditions

$$u(0,y) = 0, \qquad u(a,y) = 0, \qquad 0 < y < b$$

$$u(x,0) = h(x), \qquad u(x,b) = 0, \qquad 0 \leqslant x \leqslant a$$

Exercise 59. (3p656)

1. Find the solution $u(x,y)$ of the Laplace's equation in the rectangle $0 < x < a$, $0 < x < b$, that satisfies the boundary conditions
$$u(0,y) = 0, \qquad u(a,y) = f(y), \qquad 0 < y < b$$
$$u(x,0) = h(x), \qquad u(x,b) = 0, \qquad 0 \leqslant x \leqslant a$$

2. Find the solution if $h(x) = \left(\dfrac{x}{a}\right)^2$ and $f(y) = 1 - \dfrac{y}{b}$

Exercise 60. (5p656) Find the solution $u(r,\theta)$ of the Laplace equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

outside the circle $r = a$, that satisfies the boundary condition

$$u(a,\theta) = f(\theta), \qquad 0 \leqslant \theta < 2\pi$$

on the circle. Assume that $u(\theta, r)$ is a single valued function and is bounded for $r > a$.

Exercise 61. (6p656)

1. Find the solution $u(r,\theta)$ of the Laplace equation in a semicircular region $r < a$, $0 < \theta < \pi$, that satisfies the boundary conditions

$$u(r,0) = 0, \qquad u(r,\pi) = 0, \qquad 0 \leqslant r < a,$$
$$u(a,\theta) = f(\theta), \qquad 0 \leqslant \theta \leqslant \pi$$

   Assume that u is a single valued and bounded function in the given region.

2. Find the solution if $f(\theta) = \theta(\pi - \theta)$.

0, **Exercise 62.** (8p656)

1. Find the solution of the Laplace equation in the semi-infinite strip $0x < q$, $y > 0$, that satisfies the boundary condition

$$u(0y) = 0, \quad u(a.y) = 0, \quad y > 0,$$
$$u(x,0) = f(x), \qquad 0 \leqslant a$$

   and the additional condition that $u(x,y) \to 0$ as $y \to \infty$.

2. Find the solution if $f(x) = x(a - x)$.