

## Section 10.2 Fourier Series

**Definition:** A Fourier series is a function in the form

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{k\pi t}{L}\right) + b_k \sin\left(\frac{k\pi t}{L}\right)$$

**Remark:** A Fourier series is  $2L$  periodic.

### 1 Vector space of $2L$ -periodic functions

**Theorem:** The set of  $2L$ -periodic function is a vector space.

$$\langle f, g \rangle = \int_{-L}^L f(t), g(t) dt$$

is an inner product i. e.

- $\langle f, g \rangle = \langle g, f \rangle$ .
- $\langle af + bg, h \rangle = a\langle f, h \rangle + b\langle g, h \rangle$  for any  $2L$ -periodic functions  $f, g$ , and  $h$  and for any real numbers  $a$  and  $b$ .
- $\langle f, f \rangle \geq 0$  with  $\langle f, f \rangle = 0$  iff  $f = 0$ .

**Theorem:**

- $\langle \cos\left(\frac{k\pi t}{L}\right), \cos\left(\frac{m\pi t}{L}\right) \rangle = \begin{cases} 0 & \text{if } m \neq k \\ L & \text{if } m = k \neq 0 \\ 2L & \text{if } m = k = 0 \end{cases}$
- $\langle \sin\left(\frac{k\pi t}{L}\right), \sin\left(\frac{m\pi t}{L}\right) \rangle = \begin{cases} 0 & \text{if } m \neq k \\ L & \text{if } m = k \end{cases}$

If  $f$  is a Fourier series

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{k\pi t}{L}\right) + b_k \sin\left(\frac{k\pi t}{L}\right),$$

then, for any  $k$  in  $\mathbb{N}$ ,

- $\frac{1}{L} \langle f, \cos\left(\frac{k\pi t}{L}\right) \rangle = a_k$ .
- $\frac{1}{L} \langle f, \sin\left(\frac{k\pi t}{L}\right) \rangle = b_k$ .

**Definition:** Let  $f$  be a  $2L$ -periodic, piecewise continuous function, then

$$\bullet a_k = \frac{1}{L} \langle f, \cos \left( \frac{k\pi t}{L} \right) \rangle, \quad k = 0, 1, 2, \dots$$

$$\bullet b_k = \frac{1}{L} \langle f, \sin \left( \frac{k\pi t}{L} \right) \rangle, \quad k = 1, 2, \dots$$

and the Fourier series of  $f$ , written  $\hat{f}$  is

$$\hat{f}(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos \left( \frac{k\pi t}{L} \right) + b_k \sin \left( \frac{k\pi t}{L} \right)$$

**Exercise 1.** Given the 2-periodic function defined by

$$f(x) = \begin{cases} 0, & -1 \leq x < 0 \\ x, & 0 \leq x < 1 \end{cases}$$

- Sketch the graph of  $f$  over 3 periods.
- Find the Fourier series of  $f$ .