University of Washington

Section 7.4

Definition: A system of differential equations

$$X'(t) = A(t)X(t) + G(t)$$

- is homogeneous if $G(t) = 0$.
- is with constant coefficients if the coefficients of A are constant.

Theorem 7.1.2: If $A(t)$, and $g(t)$ are continuous on an interval I then for any initial value problem

$$X'(t) = A(t)X(t) + G(t), \qquad X(t_0) = X_0$$

with t_0 in I, there exists a unique solution to the initial value problem defined on the entire interval I.

Theorem 7.4.1, 7.4.2, 7.4.3: Let A be a $n \times n$ matrix with continuous coefficients on an interval I and consider the homogeneous system

$$X' = AX$$

- If $X_1, X_2, \cdots, X_p$ are p solutions to the system on I

  then any linear combination $X(t) = c_1 X_1 + c_2 X_2 + \cdots + c_p X_p$ is solution.

- If $X_1, X_2, \cdots, X_n$ are n linearly independent solutions to the system on I then any solution X to the system can be expressed as a linear combination of $X_1, X_2, \cdots, X_n$:

  $$X(t) = c_1 X_1 + c_2 X_2 + \cdots + c_n X_n$$

  where $c_1, c_2, \cdots, c_n$ are constants.

- If $X_1, X_2, \cdots, X_n$ are n solutions to the system on I then the Wronskian $W(X_1, X_2, \cdots, X_n) = det(X_1, X_2, \cdots, X_n)$ is either zero or never vanishes on I.
  - The Wronskian is zero if the n solutions are linearly dependent.
  - The Wronskian never vanishes if the n solutions are linearly independent

Remark:

Exercise 1. Given the system

$$X'(t) = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} X = AX.$$

1. Verify that the vectors

$$X_1(t) = \begin{pmatrix} \mathrm{e}^t \\ e^t \end{pmatrix} \qquad X_2(t) = \begin{pmatrix} \mathrm{e}^{-t} \\ 3\,\mathrm{e}^{-t} \end{pmatrix}$$

are solutions to the system.

2. Find the general solution to the differential system.

3. Solve the initial value problem

$$X' = AX \qquad X = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Exercise 2. Given the non homogeneous system

$$X' = AX + G = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & -2 \\ 2 & 2 & 1 \end{pmatrix} X + \begin{pmatrix} -9t \\ 0 \\ -18t \end{pmatrix}$$

1. Verify that $X_p = \begin{pmatrix} 5t+1 \\ 2t \\ 4t+2 \end{pmatrix}$ is a particular solution to the non homogeneous system

2. Check that the vectors

$$X_1(t) = \begin{pmatrix} e^{3t} \\ 0 \\ e^{3t} \end{pmatrix}, \qquad X_2(t) = \begin{pmatrix} -e^{3t} \\ e^{3t} \\ 0 \end{pmatrix}, \qquad X_3 = \begin{pmatrix} -e^{-3t} \\ -e^{-3t} \\ e^{-3t} \end{pmatrix}$$

are a fundamental set of solutions to the homogeneous problem.

3. Find the general solution to the non-homogeneous problem.

Theorem: The general solution $X(t)$ to a non homogeneous system

$$X'(t) = A(t)X(t) + G(t)$$

can be written in the form

$$X(t) = X_p(t) + X_h(t)$$

where

- $X_h(t)$ is the general solution to the corresponding homogeneous system.
- $X_p(t)$ is a particular solution to the non homogeneous system.