Recitation 4
Sections 7.4, 7.5, 8.1

1. A heavy rope 100 ft long, weighs 0.5lb/ft and hangs over a building. It is used to lift a 200lb container attached a the end of the rope.

   Set up an expression then find the work to pull 20 ft of the rope on top of the building.

   Answer.

   $W_{container} = 200 \cdot 20 = 4000$lb-ft.

   Let x be the height, with $x = 0$ at the top.

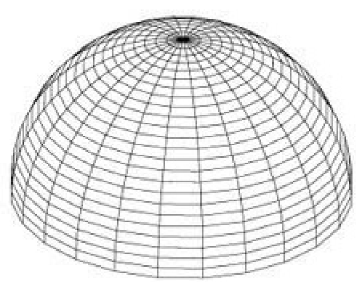
   The distance from x to the top is x. The weight of the piece of rope with length $\mathrm{d}x$ at the height x is $F = 0.5\mathrm{d}x$.

   $$W_{rope} = \int_{80}^{100} 0.5x\mathrm{d}x$$

   Expression: $W = W_{rope} + W_{container} = 4000 + \int_{80}^{100} 0.5x\mathrm{d}x$.

   $$\boxed{W = 4900\text{lb-ft.}}$$

2. A hemispherical tank is full of water with weight density ρg N/m^3 . Its radius is 2 m. Set up an integral and find the work required to pump the all water out of the top of the tank. Leave your answer in terms

   of the density ρ and g the gravitational constant.

   Answer. Let x be the height. $x = 0$ at the center of the hemisphere.

   At the height x, the horizontal slice is a disk of radius r such that $r^2 + x^2 = 4$. Therefore $r^2 = 4 - x^2$. The volume of the slice at x with thickness $\mathrm{d}x$ is $\pi r^2 \mathrm{d}x = \pi(4 - x^2)\mathrm{d}x$

   The force acting on the slice is $\rho g \pi(4 - x^2)\mathrm{d}x$

   The distance from x to the top is $2 - x$.

   The work is $W = \int_0^2 \rho g \pi(4 - x^2)(2 - x)\mathrm{d}x$

   $$\boxed{\frac{20\rho g \pi}{3}}$$

3. Find the average value of the function $f(x) = x\,\mathrm{e}^{x^2}$ over $[0, 2]$.

   Answer. $f_{ave} = \frac{1}{2}\int_0^2 x\,\mathrm{e}^{x^2}\,\mathrm{d}x$.

$$\boxed{\frac{\mathrm{e}^4 - 1}{4}}$$

4. Evaluate the integrals

   (a) $\displaystyle\int (3x + 1)\,\mathrm{e}^{-x}\,\mathrm{d}x.$

   Answer.

$$\boxed{-3x\,\mathrm{e}^{-x} - 4\,\mathrm{e}^{-x} + C}$$

   (b) $\displaystyle\int_0^{\pi/2} x^2 \cos(x)\mathrm{d}x.$

   Answer.

$$\boxed{\frac{\pi^2}{4} - 2}$$

   (c) $\displaystyle\int (3x^2 + 4x - 1)\ln(x)\mathrm{d}x.$

   Answer.

$$\boxed{(x^3 + 2x^2 - x)\ln(x) - \frac{x^3}{3} - x^2 + x + C}$$

5. Using 2 integrations by parts, evaluate $F(x) = \displaystyle\int \sin(\ln(x))\mathrm{d}x$.

   Answer.

$$u(x) = \sin(\ln x) \quad u'(x) = \frac{1}{x}\cos(\ln x)$$
$$v'(x) = 1 \quad v(x) = x$$

$$\int \sin(\ln x) = [x\sin(\ln x)] - \int \frac{x\cos(\ln(x))}{x}\mathrm{d}x$$

$$u(x) = \cos(\ln x) \quad u'(x) = -\frac{1}{x}\sin(\ln x)$$
$$v'(x) = 1 \quad v(x) = x$$

$$\int \sin(\ln x) = [x\sin(\ln x)] - [x\cos(\ln(x))] - \int \frac{x\sin(\ln x)}{x}\mathrm{d}x$$
$$F(x) = x\sin(\ln x) - x\cos(\ln x) - F(x)$$
$$\boxed{F(x) = \frac{x\sin(\ln x) - x\cos(\ln x)}{2}}$$