Recitation 5
Sections 8.2, 8.3, 8.4, 8.9

1. Find an adequate substitution and evaluate $\int \sin^4 x \cos^3 x \mathrm{d}x$.

   Answer.
   $$\sin^4 x \cos^3 x = \sin^4 x(1-\sin^2 x)\cos x$$
   With the substitution $u = \sin x, \quad \mathrm{d}u = \cos x \mathrm{d}x$,
   $$\int \sin^4 x \cos^3 x \mathrm{d}x = \int u^4 - u^6 \mathrm{d}u$$
   $$\boxed{\int \sin^4 x \cos^3 x \mathrm{d}x = \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C}$$

2. Find an adequate substitution and evaluate $\int_0^{\pi/4} \sec^4(x)\tan^6(x)\mathrm{d}x$.

   Answer. The exponent of sec is even. Let's substitute $u = \tan x \quad \mathrm{d}u = sec^2 x \mathrm{d}x$.
   $$\sec^4 x \tan^6 x = \sec^2 x(1+\tan^2 x)\tan^6 x$$
   If $x = 0$, $u = \tan 0 = 0$.
   If $x = \pi/4$, $u = \tan(\pi/4) = 1$
   $$\int_0^{\pi/4} \sec^4(x)\tan^6(x)\mathrm{d}x = \int_0^1 (1+u^2)u^6 \mathrm{d}u$$
   $$\boxed{\int_0^{\pi/4} \sec^4(x)\tan^6(x)\mathrm{d}x = \frac{1}{7} + \frac{1}{9} = \frac{16}{63}}$$

3. Find an adequate substitution and evaluate $\int_0^2 x^3\sqrt{4-x^2}\mathrm{d}x$.

   Answer. Use the substitution $x = 2\sin\theta, \quad \mathrm{d}x = 2\cos^2\theta \mathrm{d}\theta$.
   If $x = 0$, $\theta = 0$.
   If $x = 2$, $\theta = \pi/2$.
   $$\begin{aligned}\int_0^2 x^3\sqrt{4-x^2}\mathrm{d}x &= \int_0^{\pi/2} 8\sin^3\theta\sqrt{4-4\sin^2\theta}(2\cos\theta)\mathrm{d}\theta \\ &= \int_0^{\pi/2} 32\sin^3\theta\cos^2\theta \mathrm{d}\theta \\ &= 32\int_0^{\pi/2} \sin\theta(1-\cos^2\theta)\cos^2\theta \mathrm{d}\theta\end{aligned}$$

Substitute $\cos\theta = u, \quad -\sin\theta \mathrm{d}\theta = \mathrm{d}u$.

If $\theta = 0$, $u = 1$.

If $\theta = \pi/2$, $u = 0$.

$$\int_0^2 x^3\sqrt{4-x^2}\mathrm{d}x = -32\int_1^0 u^2 - u^4 \mathrm{d}u$$

$$\boxed{\frac{64}{15}}$$

4. Find a substitution and evaluate $\displaystyle\int_1^2 \sqrt{2x-x^2}\mathrm{d}x$.

Answer. Complete the square $2x - x^2 = -(x^2 - 2x + 1 - 1) = 1 - (x-1)^2$.

Substitution: $x - 1 = \sin\theta, \quad \mathrm{d}x = \cos\theta \mathrm{d}\theta$.

If $x = 1$, $\theta = 0$.

If $x = 2$, $\theta = \frac{\pi}{2}$.

$$\int_0^{\pi/2} \sqrt{1-\sin^2\theta}\cos\theta \mathrm{d}\theta = \int_0^{\pi/2} \frac{\cos(2x)+1}{2}\mathrm{d}\theta$$

$$\boxed{\frac{\pi}{4}}$$

5. Find a partial decomposition and integrate $\displaystyle\int \frac{4x^2-2x+1}{x^2(x-1)}\mathrm{d}x$.

Answer. The partial fraction decomposition is in the form

$$\frac{4x^2-2x+1}{x^2(x-1)} = \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x-1}.$$

Multiply by x^2 and set $x = 0$, $b = -1$.

Multiply by $x - 1$ and set $x = 1$, $c = 3$.

Multply by x and take the limit when x goes to ∞, $4 = a + 3$, $a = 1$.

The partial fraction decomposition is

$$\frac{4x^2-2x+1}{x^2(x-1)} = \frac{1}{x} - \frac{1}{x^2} + \frac{3}{x-1}.$$

$$\boxed{\int \frac{4x^2-2x+1}{x^2(x-1)}\mathrm{d}x = \ln x + \frac{1}{x} + 3\ln(x-1) + C.}$$

6. Find a partial fraction decomposition and integrate $\displaystyle\int \frac{3x^2-3x+1}{x(x^2+1)}\mathrm{d}x$.

Answer. The partial fraction decomposition is in the form

$$\frac{3x^2-3x+1}{x(x^2+1)} = \frac{a}{x} + \frac{bx+c}{x^2+1}$$

Multiply by x and set $x=0$, $a=1$.

Multiply by x and take the limit when x goes to ∞. $3=1+b$, $b=2$.

For $x=1$, $\frac{1}{2}=1+\frac{2+c}{2}$, $c=-3$.

The partial decomposition is

$$\frac{3x^2-3x+1}{x(x^2+1)}=\frac{1}{x}+\frac{2x-3}{x^2+1}=\frac{1}{x}+\frac{2x}{x^2+1}-3\frac{1}{x^2+1}$$

$$\boxed{\int \frac{3x^2-3x+1}{x(x^2+1)}\mathrm{d}x=\ln|x|+\ln(x^2+1)-3\,\mathrm{Arctan}(x)+C}$$

7. Determine whether the integral $\int_4^\infty \frac{1}{(x+5)^{(3/2)}}\mathrm{d}x$ is convergent or divergent. Evaluate the integral if it is convergent.

Answer.

$$\int_4^\infty \frac{1}{(x+5)^{(3/2)}}\mathrm{d}x=\lim_{t\to\infty}\int_4^t \frac{1}{(x+5)^{(3/2)}}\mathrm{d}x$$

$$\int_4^t \frac{1}{(x+5)^{(3/2)}}\mathrm{d}x=\left[\frac{-2}{\sqrt{x+5}}\right]_4^t=\frac{-2}{\sqrt{t+5}}+\frac{2}{\sqrt{9}}=\frac{2}{3}-\frac{2}{\sqrt{t+5}}.$$

$$\lim_{t\to\infty}\frac{2}{\sqrt{t+5}}=0.$$

Therefore $\lim_{t\to\infty}\int_4^t \frac{1}{(x+5)^{(3/2)}}\mathrm{d}x=\frac{2}{3}$

$$\boxed{\text{The integral is convergent and equals } \frac{2}{3}}$$