Recitation 6
Sections 8.1, 8.2, 8.3, 8.4, 8.9, 9.3, 9.4

1. Evaluate $\int_0^2 2x^3\,\mathrm{e}^{x^2}\,\mathrm{d}x$.

   Answer. Substitution: $u = x^2 \quad \mathrm{d}u = 2x\mathrm{d}x$. If $x = 0$, $u = 0$. If $x = 2$, $u = 4$.

$$\begin{aligned}\int_0^2 2x^3\,\mathrm{e}^{x^2}\,\mathrm{d}x &= \int_0^4 u\,\mathrm{e}^u\,\mathrm{d}u \\ &= [u\,\mathrm{e}^u]_0^4 - \int_0^4 \mathrm{e}^u\,\mathrm{d}u \\ &= 4\,\mathrm{e}^4 - \mathrm{e}^4 + 1\end{aligned}$$

$$\boxed{3\,\mathrm{e}^4 + 1}$$

2. Evaluate $\int_0^{\pi/4} 4\cos^2 t\sin^2 t\mathrm{d}t$.

   Answer. $\cos^2 t = \dfrac{\cos(2t)+1}{2}, \quad \sin^2 t = \dfrac{1-\cos(2t)}{2}$.

$$\begin{aligned}\int_0^{\pi/4} 4\cos^2 t\sin^2 t\mathrm{d}t &= \int_0^{\pi/4} (1+\cos(2t))(1-\cos(2t))\mathrm{d}t \\ &= \int_0^{\pi/4} 1-\cos^2(2t)\mathrm{d}t \\ &= \int_0^{\pi/4} 1-\frac{\cos(4t)+1}{2}\mathrm{d}t \\ &\left[\frac{t}{2}-\frac{\sin 4t}{8}\right]_0^{\pi/4}\end{aligned}$$

$$\boxed{\frac{\pi}{8}}$$

3. Evaluate $\int_1^{\sqrt{2}} \dfrac{\sqrt{x^2-1}}{x}\mathrm{d}x$.

   Answer. Use the substitution $x = \sec u \quad \mathrm{d}x = \sec u\tan u\mathrm{d}u$.

   If $x = 1$, $\sec u = 1$, $u = 0$.

   If $x = \sqrt{2}$, $\sec u = \sqrt{2}$, $u = \frac{\pi}{4}$.

$$\int_1^{\sqrt{2}} \frac{\sqrt{x^2-1}}{x}\mathrm{d}x = \int_0^{\pi/4} \frac{\tan u \sec u \tan u}{\sec u}\mathrm{d}u$$
$$= \int_0^{\pi/4} \tan^2 u \mathrm{d}u$$
$$= \int_0^{\pi/4} \sec^2 u - 1\mathrm{d}u$$
$$= [\tan u - u]_0^{\pi/4}$$

$$\boxed{1-\frac{\pi}{4}}$$

4. Find $\displaystyle\int_1^{\sqrt{3}} \frac{2x^2-x+5}{x(x^2+1)}\mathrm{d}x$.

Answer. The partial fraction decomposition is in the form

$$\frac{2x^2-x+5}{x(x^2+1)} = \frac{a}{x} + \frac{bx+c}{x^2+1}$$

Multiply both sides by x and set $x=0$. $a=5$.

Multiply both sides by x and take the limit when x goes to ∞.

$2=a+b$ therefore $b=-3$.

Let x be 1, $\displaystyle\frac{6}{2} = a + \frac{b+c}{2}$.

$c=-1$

$$\frac{2x^2-x+5}{x(x^2+1)} = \frac{5}{x} + \frac{-3x-1}{x^2+1} = \frac{5}{x} - \frac{3x}{x^2+1} - \frac{1}{x^2+1}$$

$$\int_1^{\sqrt{3}} \frac{2x^2-x+5}{x(x^2+1)}\mathrm{d}x = 5\ln(\sqrt{3}) - \frac{3}{2}(\ln(4)-\ln(2)) - \mathrm{Arctan}(\sqrt{3}) + \mathrm{Arctan}(1)$$

$$\boxed{5\ln(\sqrt{3}) - \frac{3}{2}\ln(2) - \frac{\pi}{12}}$$

5. Is the integral $\displaystyle\int_0^{\pi/2} \frac{\cos x}{\sqrt{\sin x}}\mathrm{d}x$ convergent or divergent? If the integral is convergent, evaluate it.

Answer. It is an improper integral of type 2 with singularity at $x=0$.

$$\int_0^{\pi/2} \frac{\cos x}{\sqrt{\sin x}}\mathrm{d}x = \lim_{t\to 0}\int_t^{\pi/2} \frac{\cos x}{\sqrt{\sin x}}\mathrm{d}x$$

Let $u=\sin x$, $\mathrm{d}u = \cos x\mathrm{d}x$

If $x=t$, $u=\sin t$. If $x=\pi/2$, $u=1$

$$\int_t^{\pi/2} \frac{\cos x}{\sqrt{\sin x}}\mathrm{d}x = \int_{\sin t}^{1} \frac{\mathrm{d}u}{\sqrt{u}} = 2\sqrt{1} - 2\sqrt{\sin t}$$

.

$$\lim_{t\to 0}\int_t^{\pi/2}\frac{\cos x}{\sqrt{\sin x}}\mathrm{d}x = 2$$

$$\boxed{\int_0^{\pi/2}\frac{\cos x}{\sqrt{\sin x}}\mathrm{d}x \text{ is convergent and equals } 2}$$

6. Let $\mathcal{C}$ be the cycloid parametrized by

$$x(t) = t - \sin t, \quad y(t) = 1 - \cos t \quad t \in [0, 2\pi].$$

Determine an integral for the length of the cycloid and evaluate it.

[*You may use the half-angle formula* $\dfrac{1-\cos t}{2} = \sin^2\left(\dfrac{t}{2}\right)$.]

Remark: The cycloid is the trajectory of a gravel stuck on your wheel when you ride your bike!

Answer. $x'(t) = 1 - \cos t, \quad y'(t) = \sin t$.

$$x'^2(t) + y'^2(t) = 1 - 2\cos t + \cos^2 t + \sin^2 t = 2(1-\cos t) = 4\sin^2\left(\frac{t}{2}\right)$$

$$\mathcal{L} = \int_0^{2\pi}\sqrt{x'^2(t)+y'^2(t)}\mathrm{d}t = \int_0^{2\pi} 2\sin\left(\frac{t}{2}\right)\mathrm{d}t = 4(-\cos\pi - (-\cos 0))$$

$$\boxed{8}$$

7. Let $\mathcal{S}$ be the surface obtained by rotating about the y axis the curve

$$y = \frac{\mathrm{e}^x + \mathrm{e}^{-x}}{2} \quad x \in [0, 1].$$

Determine an integral for the area of S and evaluate it.

Answer.

$$y'(x) = \frac{\mathrm{e}^x - \mathrm{e}^{-x}}{2}$$

$$1 + y'^2(x) = \frac{4 + \mathrm{e}^{2x} - 2 + \mathrm{e}^{-2x}}{4} = \left(\frac{\mathrm{e}^x + \mathrm{e}^{-x}}{2}\right)^2$$

$$\mathcal{A} = \int_0^1 x\sqrt{1+y'^2}\mathrm{d}x = \int_0^1 x\frac{\mathrm{e}^x + \mathrm{e}^{-x}}{2}\mathrm{d}x$$

`By integration by parts`

$$\mathcal{A} = \frac{1}{2}\left([x(\mathrm{e}^x - \mathrm{e}^{-x}]_0^1 - \int_0^1 \mathrm{e}^x - \mathrm{e}^{-x}\,\mathrm{d}x\right)$$

$$\boxed{1 - \mathrm{e}^{-1}}$$