Engineering Mathematics II
Math 152, Summer 2012

Texas A&M University

Exam 1 -keys

1. (7 points) Find the derivative of

$$f(x) = \int_0^{x^2} 3t^2 - 2t + 1 dt.$$

Answer. Let $G(X) = \int_0^X 3t^2 - 2t + 1 dt$.

By the fundamental Theorem of Calculus, $G'(X) = 3X^2 - 2X + 1$

$f(x) = G(x^2)$.

Using the chain rule, $f'(x) = 2xG'(x^2) = 2x(3(x^2)^2 - 2(x^2) + 1)$

$$\boxed{f'(x) = 6x^5 - 4x^3 + 2x.}$$

2. (7 points) Find the average value of the function $\sqrt{x}$ over $[1, 4]$

Answer. By definition $f_{ave} = \dfrac{1}{4-1}\displaystyle\int_1^4 \sqrt{x} dx$.

$$\boxed{f_{ave} = \frac{14}{9}}$$

3. (7points) A force of 40 N is required to hold a spring that has been stretched from its natural length of 30 cm to a length of 40 cm.

How much work is done in stretching the spring from 30 cm to 60 cm.

Answer. Hooke's law, $f = kx$ with $F = 40N$, $x = 0.1$m.

Therefore $k = 400$.

$W = \int_0^{0.3} kx dx = 400 \int_0^{0.3} x dx$.

$$\boxed{W = 18\text{J}}$$

4. (7 points) Evaluate the definite integral

$$\int_0^4 \frac{x}{\sqrt{1+2x}} \mathrm{d}x.$$

Answer. Use the substitution $u = 1 + 2x$, $x = \dfrac{u-1}{2}$, $\mathrm{d}u = 2\mathrm{d}x$.

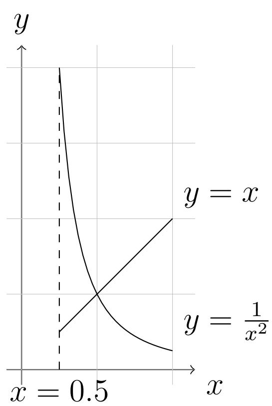
$$\boxed{\int_0^4 \frac{x}{\sqrt{1+2x}} \mathrm{d}x = \frac{10}{3}}$$

5. (7 points) Find the area between the curves $y = \dfrac{1}{x^2}$, $y = x$, $x = \dfrac{1}{2}$, and $x = 2$.

Answer. The curves cross at $x = 1$.

$$\mathcal{A} = \int_{1/2}^1 \frac{1}{x^2} - x\mathrm{d}x + \int_1^2 x - \frac{1}{x^2}\mathrm{d}x.$$

$$\boxed{\mathcal{A} = \frac{13}{8}}$$

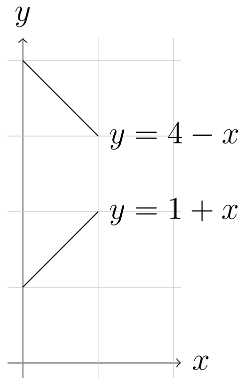


6. (7 points) Find the volume of the solid obtained by rotating about the x-axis, the area bounded by the curves $y = 1 + x$, $y = 4 - x$, $x = 0$, and $x = 1$.

Answer. Use the variable x and the washer method.

The big radius is given by $y = 4 - x$ and the small radius is given by $1 + x$.
$V = \int_0^1 \pi(4-x)^2 - \pi(1+x)^2\mathrm{d}x = \pi \int_0^1 15 - 10x\mathrm{d}x$

$$\boxed{V = 10\pi}$$



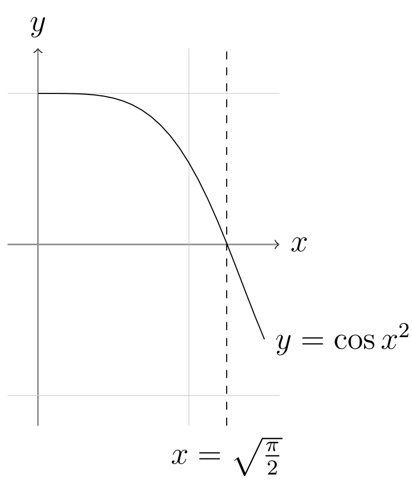
7. (7 points) Find the volume of the solid obtained by rotating about the y-axis, the area bounded by the curves $y = \cos(x^2)$, $y = 0$, $x = 0$, $x = \sqrt{\dfrac{\pi}{2}}$.

Answer. `Use the variable` x `and the cylindric shells method.`

$$V = \int_0^{\sqrt{\pi/2}} 2\pi x \cos(x^2) \mathrm{d}x \, .$$

`With the substitution` $u = x^2$, $V = \int_0^{\pi/2} \pi \cos u \mathrm{d}u \, .$

$$\boxed{V = \pi}$$



8. (12 points) Evaluate the following integrals

(a) $\displaystyle\int_0^{\pi/4} (1 + \sin(2x))^3 \cos(2x) \mathrm{d}x.$

Answer. Use the substitution $u = 1 + \sin 2x$. $\mathrm{d}u = 2\cos 2x \mathrm{d}x$
If $x = 0$, $u = 1$.
If $x = \frac{\pi}{4}$, $u = 2$.

$$\int_0^{\pi/4} (1 + \sin(2x))^3 \cos(2x)\mathrm{d}x = \int_1^2 u^3 \frac{\mathrm{d}u}{2}.$$

$$\boxed{\int_0^{\pi/4} (1 + \sin(2x))^3 \cos(2x)\mathrm{d}x = \frac{15}{8}.}$$

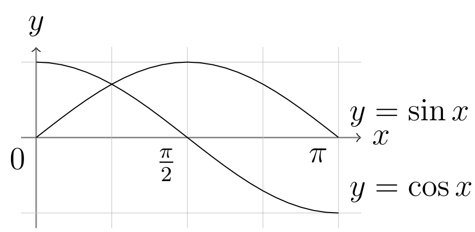
(b) $\displaystyle\int \frac{x^2+2}{x^3+6x+7}\mathrm{d}x.$

Answer. Substitution $u = x^3 + 6x + 7$. $\mathrm{d}u = (3x^2 + 6x)\mathrm{d}x = 3(x^2+2)\mathrm{d}x$

$$\int \frac{x^2+2}{x^3+6x+7}\mathrm{d}x = \int \frac{\mathrm{d}u}{3u} = \frac{1}{3}\ln|u| + C.$$

$$\boxed{\int \frac{x^2+2}{x^3+6x+7}\mathrm{d}x = \frac{1}{3}\ln\left|3x^2+6x\right| + C}$$

9. (12 points) Find the area between the curves $y = \cos x$, and $y = \sin x$ over $[0, \pi]$.



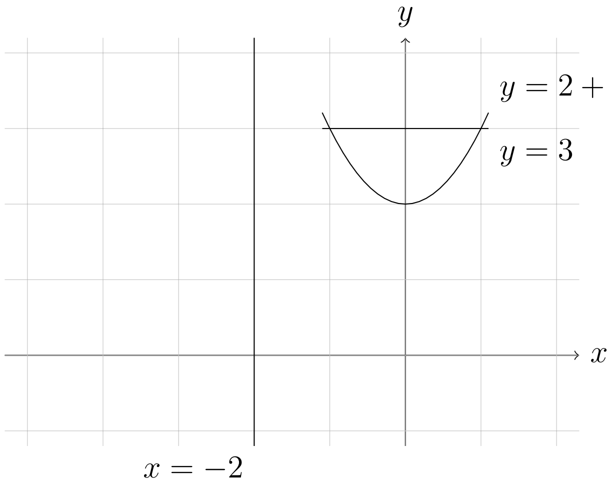
Answer. The curves intersect when $\cos(x) = \sin(x)$.

It happens when $x = \pi/4 + k\pi$ $(k \in \mathbb{Z})$. Over $[0, \pi]$, The curves intersect at $x = \pi/4$.

$$A = \int_0^{\pi/4} \cos x - \sin x \mathrm{d}x + \int_{\pi/4}^{\pi} \sin x - \cos x \mathrm{d}x = (\sqrt{2} - 1) + (\sqrt{2} + 1).$$

$$\boxed{A = 2\sqrt{2}}$$

10. (14 points) Find the volume of the solid obtained by the rotation about the line $x = -2$ of the region bounded by the curves $y = 2 + x^2$ and $y = 3$.



Answer. Use the cylindric shells method with the variable x.

The 2 curves intersect at $x = 1$ and $x = -1$. (Solve $2 + x^2 = 3$)

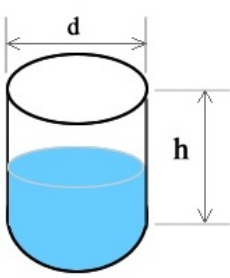
The height of the cylinders are $top\_curve - bottom\_curve = 3 - (2 + x^2) = 1 - x^2$.

The radius of the cylinder is $x - (-2) = x + 2$.

Therefore $V = \int_1^3 2\pi(x + 2)(1 - x^2)\mathrm{d}x$

$$\boxed{V = \frac{16\pi}{3}}$$

11. (13 points) A cylindrical tank has height $h = 3m$. Its diameter is $d = 4m$. The tank is **half full** of liquid with density ρ kg/m^3. What is the work required to pump all the liquid out of the top of the tank. Leave your answer in terms of ρ the density and g the gravitational constant.



Answer.

$\boxed{\text{see solution on Piazza}}$