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Keys for exam 2

1. (7 points) Evaluate
$$\int \frac{2\ln(t)}{t^3}\mathrm{d}t.$$

Answer. `integration by parts`

$$u(t) = \ln(t) \qquad u'(t) = \frac{\mathrm{d}t}{t}$$
$$v'(t) = \frac{2}{t^3} \qquad v(t) = \frac{-1}{t^2}4$$
$$\int \frac{2\ln(t)}{t^3}\mathrm{d}t. = \frac{-\ln(t)}{t^2} + \int \frac{1}{t^3}\mathrm{d}t$$
$$\boxed{\int \frac{2\ln(t)}{t^3}\mathrm{d}t = -\frac{(2\ln(t)+1)}{2t^2} + C}$$

2. (7 points) Evaluate $\int_0^{\pi/2} \sin^3 t \cos^5 t \mathrm{d}t$.

Answer. `Odd exponent for` $\sin$, `therefore substitution` $u = \cos t$.

$\mathrm{d}u = -\sin t\mathrm{d}t$ `and` $sin^2 t = 1 - \cos^2 t = 1 - u^2$.

`If` $t = 0$, $u = 1$.

`If` $t = \frac{\pi}{2}$, $u = 0$.

$$\int_0^{\pi/2} \sin^3 t \cos^5 t\mathrm{d}t = \int_1^0 (1-u^2)u^5(-\mathrm{d}u).$$

$$\boxed{\int_0^{\pi/2} \sin^3 t \cos^5 t\mathrm{d}t = \frac{1}{24}}$$

3. (7 points) After substitution the integral $\int_{\sqrt{3}/3}^{1} x^4\sqrt{x^2+1}\mathrm{d}x$ is equivalent to

Answer. `Substitution` $x = \tan u$, $\mathrm{d}x = (\sec^2 u)\mathrm{d}u$.

`If` $x = \frac{\sqrt{3}}{3}$, $\tan u = \frac{\sqrt{3}}{3}$, $u = \frac{\pi}{6}$.

If $x = 1$, $\tan u = 1$, $u = \dfrac{\pi}{4}$.

$$\boxed{\int_{\sqrt{3}/3}^{1} x^4\sqrt{x^2+1}\mathrm{d}x = \int_{\pi/6}^{\pi/4} \tan^4 u \sec^3 u \mathrm{d}u}$$

4. (7 points) $\displaystyle\int \frac{5x^2-8x+5}{x^2(x-1)}\mathrm{d}x$ equals

   Answer. Partial fraction decomposition: $\dfrac{5x^2-8x+5}{x^2(x-1)} = \dfrac{-5}{x^2} + \dfrac{3}{x} + \dfrac{2}{(x-1)}$.

   Integrate the decomposition:

$$\boxed{\int \frac{5x^2-8x+5}{x^2(x-1)}\mathrm{d}x = \frac{5}{x} + 2\ln|x-1| + 3\ln|x| + C}$$

5. (7 points) The improper integral $\displaystyle\int_1^{\infty} \frac{2}{x^5+\sqrt{x}}\mathrm{d}x$

   Answer. For $x > 1$, $x^5 + \sqrt{x} > x^5$, therefore $\dfrac{2}{x^5+\sqrt{x}} < \dfrac{2}{x^5}$.

   $\displaystyle\int_1^{\infty} \frac{2}{x^5}\mathrm{d}x$ is convergent to $\dfrac{1}{2}$.

$$\boxed{\int_1^{\infty} \frac{2}{x^5+\sqrt{x}}\mathrm{d}x \text{ converges by comparison to } \int_1^{\infty} \frac{2}{x^5}\mathrm{d}x}$$

   Remark: $\dfrac{2}{\sqrt{x}+x^5} < \dfrac{2}{\sqrt{x}}$ and $\displaystyle\int_1^{\infty} \frac{2}{\sqrt{x}}\mathrm{d}x$ is divergent. The comparison theorem doesn't tell us anything in this case.

6. (7 points) The area of the surface obtained by the rotation of the curve $y = x^2 + 1$ about the x-axis for $x \in [0, 2]$ is given by the following integral

   Answer. The variable is x.

   $y'(x) = 2x$ and $\sqrt{1 + y'^2(x)} = \sqrt{1+4x^2}$.

   The radius is $y = (1 + x^2)$.

$$\boxed{\mathcal{A} = \int_0^2 2\pi(x^2+1)\sqrt{1+4x^2}\mathrm{d}x}$$

7. (7 points) Find the length of the parametrized curve

$$x(t) = \cos^3 t, \quad y(t) = \sin^3 t, \quad t \in [0, \pi/4].$$

Answer. $x'(t) = -3\sin t\cos^2 t$, $y'(t) = 3\cos t\sin^2 t$.

$$\begin{aligned} x'^2(t) + y'^2(t) &= 9\sin^2 t\cos^4 t + 9\cos^2 t\sin^4 t \\ &= 9\sin^2 t\cos^2 t(\cos^2 t + \sin^2 t) \\ &= 9\sin^2 t\cos^2 t. \end{aligned}$$

$\sqrt{x'^2(t) + y'^2(t)} = 3\sin t\cos t$.

$l = \displaystyle\int_0^{\pi/4} \sqrt{x'^2(t) + y'^2(t)}\mathrm{d}t = \int_0^{\pi/4} 3\sin t\cos t\mathrm{d}t$

$$\boxed{l = \frac{3}{4}}$$

8. (15 points) Consider the curve $\mathcal{C}$ given by

$$x = \ln(\cos y) \quad y \in [0, \pi/4]$$

(a) Calculate the length of the curve $\mathcal{C}$.

Answer. `Chain rule:` $x'(y) = \dfrac{-\sin y}{\cos y}$.

$\sqrt{1 + x'^2(y)} = \sqrt{1 + \tan^2 y} = \sec y$

$l = \displaystyle\int_0^{\pi/4} \sec y\mathrm{d}y = [\ln(\sec y + \tan y)]_0^{\pi/4}$.

$$\boxed{l = \ln(1 + \sqrt{2})}$$

(b) Set up an integral for the area of the surface obtained by rotating the curve $\mathcal{C}$ about the x-axis.

Answer.

$$\boxed{\mathcal{A} = \int_0^{\pi/4} 2\pi y\sqrt{1 + x'^2(y)}\mathrm{d}y = \int_0^{\pi/4} 2\pi y\sec y\mathrm{d}y}$$

Do not integrate.

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9. (17 points) Calculate each integral:

(a)

$$\int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4-x^2}}\mathrm{d}x$$

Answer. Substitution $x = 2\sin u$. $\mathrm{d}x = 2\cos u\mathrm{d}u$.
If $x = 0$, $2\sin u = 0$, $u = 0$.
If $x = \sqrt{2}$, $2\sin u = \sqrt{2}$, $\sin u = \dfrac{\sqrt{2}}{2}$.

$$\begin{aligned}\int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4-x^2}}\mathrm{d}x &= \int_0^{\pi/4} \frac{4\sin^2 u}{\sqrt{4-4\sin^2 u}} 2\cos u\mathrm{d}u \\ &= \int_0^{\pi/4} 4\sin^2 u\mathrm{d}u \\ &= \int_0^{\pi/4} 2(1-\cos(2u))\mathrm{d}u\end{aligned}$$

$$\boxed{\int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4-x^2}}\mathrm{d}x = \frac{\pi}{2} - 1}$$

(b)

$$\int \frac{5x^2-4x+3}{(x-1)(x^2+1)}\mathrm{d}x$$

Answer. Partial fraction decomposition

$$\frac{5x^2-4x+3}{(x-1)(x^2+1)} = \frac{3x-1}{x^2+1} + \frac{2}{x-1} = \frac{3x}{x^2+1} - \frac{1}{x^2+1} + \frac{2}{x-1}$$

$$\boxed{\int \frac{5x^2-4x+3}{(x-1)(x^2+1)}\mathrm{d}x = \frac{3}{2}\ln(x^2+1) - \operatorname{Arctan} x + 2\ln|x-1| + C}$$

10. (19 points)

(a) Determine whether the improper integral $\displaystyle\int_1^5 \frac{1}{\sqrt{x-1}}\mathrm{d}x$ converges and evaluate it if it does.

Answer. The integral is improper of type 2 at $x = 1$.

$$\int_1^5 \frac{1}{\sqrt{x-1}}\mathrm{d}x = \lim_{t\to 1}\int_t^5 \frac{1}{\sqrt{x-1}}\mathrm{d}x = \lim_{t\to 1}(2\sqrt{5-1} - 2\sqrt{t-1})$$

$$\boxed{\int_1^5 \frac{1}{\sqrt{x-1}}\mathrm{d}x \text{ is convergent, equals } 4}$$

(b) Determine whether the improper integral $\displaystyle\int_5^\infty \frac{1}{\sqrt{x-1}}\mathrm{d}x$ converges and evaluate it if it does.

Answer. `Improper integral of type 1.`

$$\int_5^\infty \frac{1}{\sqrt{x-1}}\mathrm{d}x = \lim_{t\to\infty}\int_5^t \frac{1}{\sqrt{x-1}}\mathrm{d}x = \lim_{t\to\infty} 2\sqrt{t-1}-4=\infty$$

$$\boxed{\int_5^\infty \frac{1}{\sqrt{x-1}}\mathrm{d}x \text{ is divergent.}}$$

(c) Is the integral $\displaystyle\int_1^\infty \frac{1}{\sqrt{x-1}}\mathrm{d}x$ convergent or divergent? Why?

Answer. `Improper of type 1 at` ∞ `and improper of type 2 at` 1.

`Split the integral to isolate the singularities`

$$\int_1^\infty \frac{1}{\sqrt{x-1}}\mathrm{d}x = \int_5^\infty \frac{1}{\sqrt{x-1}}\mathrm{d}x + \int_1^5 \frac{1}{\sqrt{x-1}}\mathrm{d}x\,.$$

$\displaystyle\int_5^\infty \frac{1}{\sqrt{x-1}}\mathrm{d}x$ `is divergent therefore`

$$\boxed{\int_1^\infty \frac{1}{\sqrt{x-1}}\mathrm{d}x \text{ is divergent.}}$$