Texas A&M University

Review for Exam 1
Sections 6.4-7.5

1 Section 6.4

Exercise 1. Find the derivative of $\displaystyle\int_0^{\cos x} t^2 - 5\mathrm{d}t$

Answer.
Let $g(x) = \int_0^{\cos x} t^2 - 5\mathrm{d}t$ and $F(X) = \int_0^X t^2 - 5\mathrm{d}t$.
The fundamental theorem of calculus implies $F'(X) = X^2 - 5$.
Remark that $g(x) = F(\cos x)$.
$g'(x) = (-\sin x)F'(\cos x)$ (Chain rule)

$$\boxed{g'(x) = (-\sin x)(\cos^2 x - 5)}$$

Exercise 2. Find $\displaystyle\int_{-1}^{2} 1 - |x|\,\mathrm{d}x$.

Answer.
For $x \in [-1, 0]$, $|x| = -x$, therefore $1 - |x| = 1 + x$.
For $x \in [0, 2]$, $|x| = x$, therefore $1 - |x| = 1 - x$.
$\displaystyle\int_{-1}^{2} 1 - |x|\,\mathrm{d}x = \int_{-1}^{0} 1 + x\mathrm{d}x + \int_0^2 1 - x\mathrm{d}x$

$$\boxed{\frac{1}{2}}$$

Exercise 3. Find the derivative of $f(x) = \displaystyle\int_{-x}^{x} t^2 + 1\ dt$ at $x = 1$, at $x = 0$.

Answer.

$$f(x) = -\int_0^{-x} t^2 + 1\mathrm{d}t + \int_0^x t^2 + 1\mathrm{d}t$$

$$f'(x) = [-(-1)((-x)^2 + 1)] + [x^2 + 1] = 2x^2 + 2$$

$$\boxed{f'(1) = 4, \quad f'(0) = 2}$$

2 Section 6.5

Exercise 4. Find the following integrals

1. Find $\displaystyle\int 4x\sqrt{2x-3}\mathrm{d}x$.

   Answer. `Substitution` $u = 2x-3$. $\mathrm{d}u = 2\mathrm{d}x$. $x = \dfrac{u+3}{2}$.

$$\begin{aligned}\int 4x\sqrt{2x-3}\mathrm{d}x &= \int (u+3)\sqrt{u}\mathrm{d}u \\ &= \int u^{(3/2)} + 3u^{(1/2)}\mathrm{d}u \\ &= \frac{2u^{(5/2)}}{5} + 2u^{(3/2)} + C\end{aligned}$$

$$\boxed{\frac{2(2x-3)^{(5/2)}}{5} + 2(2x-3)^{(3/2)} + C}$$

2. $\displaystyle\int \cos x(\sin x + 1)^2\mathrm{d}x$.

   Answer. `Subtitution` $u = 1 + \sin x$

$$\boxed{\frac{1}{3}(1+\sin x)^3 + C}$$

3. $\displaystyle\int 6x\,\mathrm{e}^{x^2}\,\mathrm{d}x$.

   Answer. `Substitution` $u = x^2$, $\mathrm{d}u = 2x\mathrm{d}x$.

$$\boxed{3\,\mathrm{e}^{x^2} + C}$$

4. $\displaystyle\int_{\sqrt{2}/2}^{1/2} \frac{1}{\sqrt{1-x^2}}\mathrm{d}x$.

   Answer. `No substitution needed,` $I = \arcsin\left(\frac{1}{2}\right) - \arcsin\left(\frac{\sqrt{2}}{2}\right)$.

$$\boxed{\frac{\pi}{6} - \frac{\pi}{4} = -\frac{\pi}{12}}$$

5. $\displaystyle\int_0^{\pi} \sin^{10} x\cos^3 x\mathrm{d}x$.

   Answer. $I = \int_0^{\pi/2} \sin^{10} x\cos^3 x\mathrm{d}x + \int_{\pi/2}^{\pi} \sin^{10} x\cos^3 x\mathrm{d}x$.

   `Substitution` $u = \pi - x$ `in the second integral,`

$$\begin{aligned} I &= \int_0^{\pi/2} \sin^{10} x\cos^3 x\mathrm{d}x + \int_{\pi/2}^{0} \sin^{10} u(-\cos u)^3(-\mathrm{d}u) \\ &= \int_0^{\pi/2} \sin^{10} x\cos^3 x\mathrm{d}x - \int_0^{\pi/2} \sin^{10} u\cos^3 u\mathrm{d}u\end{aligned}$$

$$\boxed{0}$$

6. $\displaystyle\int_0^2 6x^2\sqrt{x^3+1}\mathrm{d}x.$

Answer. `Substitution` $u = x^3 + 1$, $I = \int_1^9 2\sqrt{u}\mathrm{d}u$

$$\boxed{I = \frac{104}{3}}$$

7. $\displaystyle\int_0^{\pi/12} \tan 3x\mathrm{d}x.$

Answer. `Substitution` $u = \cos 3x$, $\mathrm{d}u = 3\sin 3x\mathrm{d}x$

$$\boxed{\frac{1}{6}\ln(2)}$$

8. $\displaystyle\int (2+3\,\mathrm{e}^x)^7\,\mathrm{e}^x\,\mathrm{d}x.$

Answer. `Substitution` $u = 2 + 3\,\mathrm{e}^x$,

$$\boxed{\frac{(2+3\,\mathrm{e}^x)^8}{24} + C}$$

9. $\displaystyle\int \cos(5x-6)\mathrm{d}x.$

Answer. `Substitution` $u = 5x - 6$,

$$\boxed{\frac{\sin(5x-6)}{5} + C}$$

10. $\displaystyle\int_0^3 \mathrm{e}^{x^3+1}\,x^2\mathrm{d}x.$

Answer. `Substitution` $u = x^3 + 1$, $I = \dfrac{1}{3}\displaystyle\int_1^{28} e^u\mathrm{d}u$,

$$\boxed{\frac{\mathrm{e}^{28} - \mathrm{e}}{3}}$$

11. $\displaystyle\int_0^{\pi/2} \frac{\sin x}{2+\cos x}\mathrm{d}x.$

Answer. `Substitution` $u = 2 + \cos x$, $\mathrm{d}u = -\sin x\mathrm{d}x$.

`If` $x = 0$, $u = 3$. `If` $x = \dfrac{\pi}{2}$, $u = 2$.

$I = \displaystyle\int_3^2 \frac{-1}{u}\mathrm{d}u.$

$$\boxed{\ln\left(\frac{3}{2}\right)}$$

12. $\displaystyle\int_0^1 \frac{x}{\sqrt{1-x^2}}\mathrm{d}x$

Answer. `Substitution` $u = 1 - x^2$

$$\boxed{1}$$

13. $\int t^2\sqrt{t^3+5}\mathrm{d}t.$

Answer. Substitution $u = t^3 + 5$,

$$\boxed{\frac{2}{9}\left(t^3+5\right)^{(3/2)}+C}$$

14. $\int_2^3 \frac{x^2}{x^3-1}\mathrm{d}x.$

Answer. Substitution $u = x^3 - 1$,

$$\boxed{\frac{1}{3}\ln\left(\frac{26}{7}\right)}$$

15. $\int 18x^2(6x^3+7)^{(1/2)}\mathrm{d}x.$

Answer. Substitution $u = 6x^3 + 7$,

$$\boxed{\frac{2}{3}\left(6x^3+7\right)^{(3/2)}+C}$$

16. $\int \frac{16u+7}{8u^2+7u}\mathrm{d}u.$

Answer. Substitution $u = 8u^2 + 7u$,

$$\boxed{\ln\left|8u^2+7u\right|+C}$$

17. $\int_{1/\pi}^{(\mathrm{e}^4)/\pi} \frac{\ln(\pi t)}{t}\mathrm{d}t.$

Answer. Remark: you may use that $ln(\pi t) = \ln(\pi) + \ln(t)$,

Substitution $u = \ln(\pi t)$. $I = \int_1^4 u\mathrm{d}u$,

$$\boxed{\frac{15}{2}}$$

18. $\int_1^3 x\sqrt{x-1}\mathrm{d}x.$

Answer. Substitution $u = x - 1$, $I = \int_0^2 (u+1)\sqrt{u}\mathrm{d}u$

$$\boxed{\frac{44}{15}\sqrt{2}}$$

19. $\int_0^5 x(x-3)^6\mathrm{d}x.$

Answer. Substitution $u = x - 3$,

$$I = \int_{-3}^2 (u+3)u^6\mathrm{d}u = \left[\frac{x^8}{8}+\frac{3x^7}{7}\right]_{-3}^2.$$

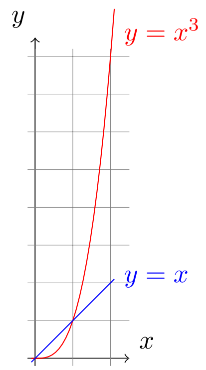
$$\boxed{\frac{11425}{56}}$$

3 Section 7.1

Exercise 5. Find the area between the curves

1. $y = x^3$ and $y = x$ over $[0, 2]$.

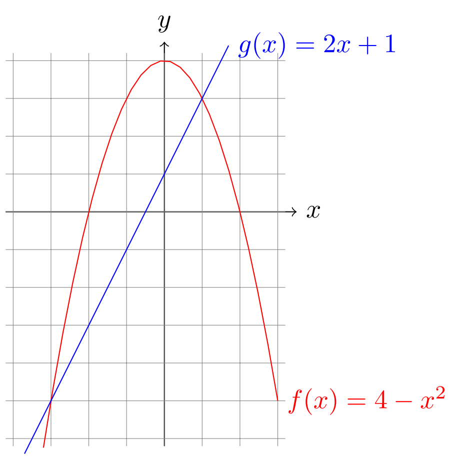
   Answer.

   Curves intersect at $x = 1$. $\mathcal{A} = \int_0^1 x - x^3 \mathrm{d}x + \int_1^2 x^3 - x \mathrm{d}x$

   $$\boxed{\frac{5}{2}}$$

2. $f(x) = 4 - x^2$ and $g(x) = 2x + 1$

   Answer.

   - Find the coordinates of the intersection points by solving $y = g(x) = f(x)$.
     $4 - x^2 = 2x + 1$ iff $0 = x^2 + 2x - 3$.
     Using the quadratic formula, 2 solutions : $x = 1$ and $x = -3$.

- Evaluate the area:

$$\begin{aligned} A &= \int_{-3}^{1} top\_function - bottom\_function \\ &= \int_{-3}^{1} (4-x^2) - (2x+1)\mathrm{d}x \\ &= \left[3x - x^2 - \frac{x^3}{3}\right]_{-3}^{1} \end{aligned}$$

$$\boxed{\frac{32}{3}}$$

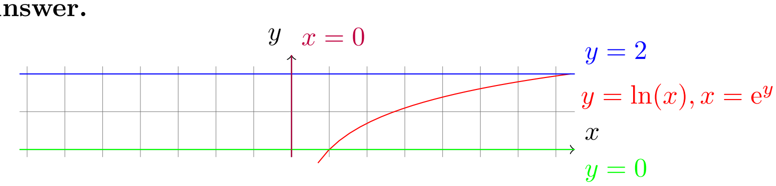
3. $f(x) = x^3$ and $g(x) = x^2$ between $x = 0$ and $x = 4$.

Answer. The curves intersect at $x = 1$. $A = \int_0^1 x^2 - x^3 \mathrm{d}x + \int_1^4 x^3 - x^2 \mathrm{d}x$

$$\boxed{A = \frac{257}{6}}$$

4 Sections 7.2, 7.3

Exercise 6. (Fall 2011) Find the volume of the solid formed by rotating the region bounded by $x = 0$, $y = \ln x$, $y = 0$, $y = 2$ about the y-axis.
Answer.



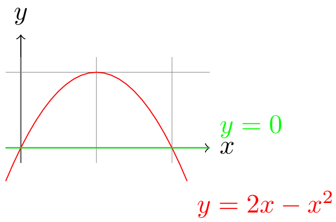
- Washer method: $V = \int_0^2 \pi(\mathrm{e}^y)^2 \mathrm{d}y = \int_0^2 \pi\, \mathrm{e}^{2y}\, \mathrm{d}y = \frac{\pi(\mathrm{e}^4 - 1)}{2}$.
- The cylindric shells method gives:

  $$V = \int_1^{\mathrm{e}^2} 2\pi x(2 - \ln(x))\mathrm{d}x = \int_1^{\mathrm{e}^2} 4\pi x - 2\pi x \ln(x)\mathrm{d}x.$$

  Use integration by part to evaluate $\int_6^{\mathrm{e}^2} x\ln(x)\mathrm{d}x$.

$$\boxed{V = \frac{\pi(\mathrm{e}^4 - 1)}{2}}$$

Exercise 7. The region bounded by $y = 2x - x^2$ and $y = 0$ is revolved around the y-axis. Find the volume.
Answer.



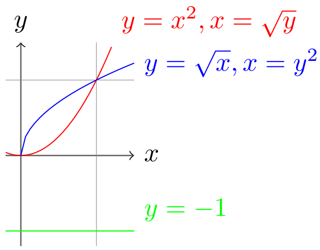
Using the cylindric shells method,

$$V = \int_0^2 2\pi x(2x - x^2)\mathrm{d}x = \int_0^2 4\pi x^2 - 2\pi x^3 \mathrm{d}x$$

$$\boxed{\frac{8\pi}{3}}$$

Exercise 8. Using cylindric shells, set up an integral for the volume of the solid formed by rotating the region bounded by $y = \sqrt{x}$ and $y = x^2$ about the line $y = -1$.

Answer.

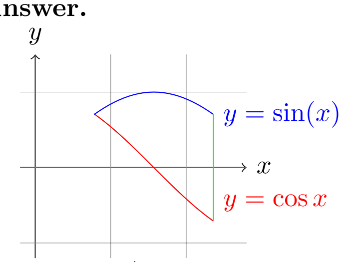


- $V = \int_0^1 2\pi(y - (-1))(\sqrt{y} - y^2)\mathrm{d}y = \int_0^1 2\pi(y+1))(\sqrt{y} - y^2)\mathrm{d}y$.
- Use the washer method, $V = \int_0^1 \pi(\sqrt{x} - (-1))^2 - \pi(x^2 - (-1))^2\mathrm{d}x = \pi \int_0^1 x + 2\sqrt{x} + 1 - x^4 - 2x^2 - 1\mathrm{d}x$
  $V = \pi\left(\frac{1}{2} + \frac{4}{3} - \frac{1}{5} - \frac{2}{3}\right)$

$$\boxed{\int_0^1 2\pi(y+1))(\sqrt{y} - y^2)\mathrm{d}y = \frac{29\pi}{30}}$$

Exercise 9. (Fall 2011) Find the volume of the solid whose base is the area enclosed by $y = \sin x$ and $y = \cos x$ from $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$ with cross sections perpendicular to the x-axis that are squares.

Answer.



$V = \int_{\pi/4}^{3\pi/4} A_{square}(x)\mathrm{d}x$.

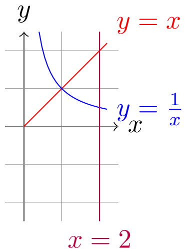
At a given x, the side of the square measures $\sin x - \cos x$.

$$A_{square}(x) = (\sin x - \cos x)^2 = 1 + 2\sin x \cos x$$

$$V = \int_{\pi/4}^{3\pi/4} 1 + 2\sin x \cos x \mathrm{d}x$$

$$\boxed{V = \frac{\pi}{2}}$$

Exercise 10. Find the volume of the solid obtained by rotating the region bounded by the curves $y = \frac{1}{x}$, $y = x$, and $x = 2$ about the x-axis.



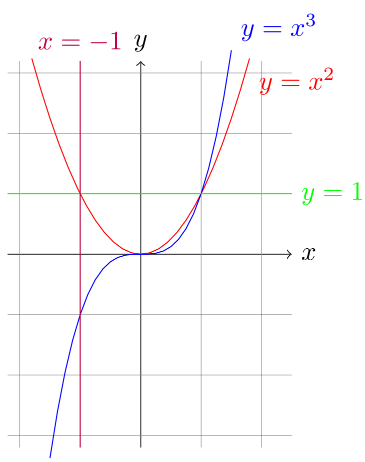
Answer. `Using the washer method:`

$$V = \pi \int_1^2 x^2 - \left(\frac{1}{x}\right)^2 \mathrm{d}x$$

$$\boxed{V = \frac{11\pi}{6}}$$

Exercise 11. Find the volume of the solid obtained by rotating the region bounded by the curves $y = x^3$ and $y = x^2$

1. about the line $y = 1$.



Answer. `Using the washer method,`

$$V = \pi \int_0^1 (1 - x^3)^2 - (1 - x^2)\mathrm{d}x$$

$$\boxed{\frac{23\pi}{210}}$$

2. about the line $x = -1$.

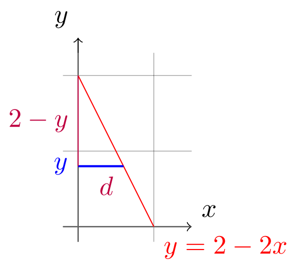
Answer. `Using the cylindric shells method,`

$V = \int_0^1 2\pi(x - (-1))(x^2 - x^3)\mathrm{d}x$.

$$\boxed{\frac{4\pi}{15}}$$

Exercise 12. Let $\mathcal{S}$ be a solid whose base is the triangle with vertices $(0, 0)$, $(1, 0)$, and $(0, 2)$, and whose cross sections perpendicular to the y-axis are semicircles. Compute the volume of $\mathcal{S}$.

Answer.



Above each horizontal segment (blue) $y = cte$ is a semi-circle whose diameter is the blue segment. The segment are parametrized by y therefore the variable of integration is y.
$V = \int_0^2 \mathcal{A}_y \mathrm{d}y$ where A_y is the area of the semi-circle at $y = cte$. The area is $\frac{\pi r^2}{2} = \frac{\pi d^2}{8}$ where d is the length of the blue segment.
Using the similar triangle equalities,

$$\frac{d}{1} = \frac{2-y}{2}$$

Therefore $\mathcal{A}_y = \frac{\pi(2-y)^2}{32}$.
$V = \int_0^2 \frac{\pi(2-y)^2}{32}\mathrm{d}y$.

$$\boxed{V = \frac{\pi}{12}}$$

5 Section 7.4

Exercise 13. (4p448) When a particle is a a distance x meters from the origin, a force of $\cos(\pi x/3)$ Newtons acts on it. How much work is done in moving the particle from $x = 1$ to $x = 2$?
Answer. $W = \int_1^2 F(x)\mathrm{d}x = \int_1^2 \cos(\pi x/3) = \left[\frac{3\sin(\pi x/3)}{\pi}\right]_1^2 = \frac{3(\sqrt{3}-\sqrt{3})}{2\pi}$

$$\boxed{W = 0J}$$

Exercise 14. A 15N weight is suspended vertically at the end of a 30m long rope. The rope weighs 6N. How much work is required to pull the weight to the top?
Answer. $W = W_{rope} + W_{weight}$.
$W_{weight} = F \cdot d = 6 * 30 = 180J$. $W_{rope} = \int_0^{30} x\frac{15\mathrm{d}x}{30} = 225J$.

$$\boxed{W = 405J}$$

Exercise 15. A rope 20 feet long weighing 2 pounds per foot supports a 160lb weight on the side of the building. How much work in (ft-lb) is required to pull the weight to the top of the building?

Answer. $W = W_{rope} + W_{weight}$.

$W_{weight} = F \cdot d = 160 * 20 = 3200\texttt{ft-lb}$.

Let x be the distance to the top. $x = 0$ at the top of the building. $x = 20$ at the bottom of the rope.

$W_{rope} = \int_0^{20} 2x\mathrm{d}x = 400\texttt{ft-lb}$

$$\boxed{W = 3600\text{ft-lb}}$$

Exercise 16. (8p448) If the work to stretch a spring 1 ft beyond its natural length is 12 ft-lb, how much work is needed to stretch it 9 in. beyond its natural length?

Answer. Hooke's law $F = kx$ where x is the length beyond its natural length.

$$W_{1\texttt{ft}} = 12 \texttt{ ft-lb} = \int_0^1 kx\mathrm{d}x = \frac{k}{2}\texttt{ft-lb}$$

Therefore $k = 24$.

Remark: 9in=0.75 ft.

$$\boxed{W_{0.75} = \frac{27}{4}\text{ft-lb}}$$

Exercise 17. A force of 10lb is required to hold a spring stretched 1/3 ft beyond its natural length. How much work is done in stretching it from its natural length to 1/2 ft beyond its natural length?

Answer. Hooke's law $F = kx$. Here $10 = k\frac{1}{3}$, therefore $k = 30$.

$$W = \int_0^{0.5} kx\mathrm{d}x = \int_0^{0.5} 30x\mathrm{d}x$$

$$\boxed{W = \frac{15}{4}\text{ft-lb}}$$

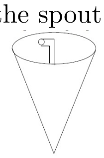
Exercise 18.

(10 points) A cylindrical stock tank has height $h = \frac{2}{3}$ m. The diameter is $d = 1$ m; illustration below. The tank is full of liquid (density $= \rho$ kg/m^3). What is the work required to pump all the liquid out of the top of the stock tank? (Leave your answer in terms of ρ and g the gravitational constant.)

Answer.

see past exam

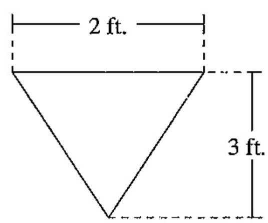
Exercise 19. The conical tank shown below is 3 feet tall (not including the spout), has a 2 foot radius at the top, is full of water (density ρg), and has a 1 foot tall spout. Find the work required to pump all the water out of the spout. Leave your answer in terms of ρ and g.



Answer.

see past exam

Exercise 20.
Consider a trough full of liquid with weight density ρg, where the trough is 8 ft. long and its cross-section is given by the figure



Calculate the work needed to pump all of the fluid over the top. (10 points)

Answer.

see past exam

6 Section 7.5

Exercise 21. Find the average value of the following functions

1. $g(x) = \dfrac{3}{x}$ over $[1, 3]$.

   Answer. $g_{ave} = \dfrac{1}{(3-1)} \displaystyle\int_1^3 \frac{3}{x} \mathrm{d}x$

   $$g_{ave} = \frac{3\ln(3)}{2}$$

2. $j(x) = x^2$ over $[1, 4]$.

   Answer. $j_{ave} = \dfrac{1}{(4-1)} \displaystyle\int_1^4 x^2 \mathrm{d}x$

   $$j_{ave} = 7$$

3. (Fall 2011) $f(x) = \cos^2 x \sin(2x)$ over the interval $\left[0, \frac{\pi}{2}\right]$.

   Answer. $f_{ave} = \frac{2}{\pi}\int_0^{\pi/2} \cos^2 x(2\sin x \cos x)\mathrm{d}x\,.$

$$\boxed{f_{ave} = \frac{1}{\pi}}$$

4. $f(x) = \dfrac{x}{\sqrt{x+1}}$ on the interval $[0, 3]$.

   Answer. $f_{ave} = \frac{1}{3}\int_0^{3} \frac{x}{\sqrt{x+1}}\mathrm{d}x\,.$ `Substitution` $u = x + 1\,.$

$$\boxed{f_{ave} = \frac{8}{9}}$$