Texas A&M University

Review for Exam 2
Sections 8.1-9.4

1 Section 8.1

Exercise 1.

1. $\displaystyle\int x\cos(3x)\mathrm{d}x.$

   Answer. `By integration by parts`

$$\boxed{\int x\cos 3x\mathrm{d}x = \frac{x\sin 3x}{3} + \frac{\cos 3x}{9} + C}$$

2. $\displaystyle\int (x^2+x)\sin x\mathrm{d}x.$

   Answer. `2 integrations by parts`

$$\boxed{-(x^2+x)\cos x + (2x+1)\sin x + 2\cos x + C}$$

3. $\displaystyle\int x\cos^3 x\mathrm{d}x.$

   Answer. `By integration by parts,`

$$\begin{aligned}\int x\cos^3 x\mathrm{d}x &= \int x(1-\sin^2 x)\cos x\mathrm{d}x \\ &= \left[x\left(\sin x - \frac{\sin^3 x}{3}\right)\right] - \int \sin x - \frac{\sin^3 x}{3}\mathrm{d}x \\ &= \left[x\left(\sin x - \frac{\sin^3 x}{3}\right)\right] + \cos x + \frac{1}{3}\int (1-\cos^2 x)\sin x\mathrm{d}x\end{aligned}$$

$$\boxed{x\left(\sin x - \frac{\sin^3 x}{3}\right) + \frac{2}{3}\cos x + \frac{1}{9}\cos^3 x + C}$$

4. $\displaystyle\int x^2\,\mathrm{e}^{-x}\,\mathrm{d}x.$

   Answer. `2 integrations by parts`

$$\boxed{\int x^2\,\mathrm{e}^{-x}\,\mathrm{d}x = (-x^2-2x-2)\,\mathrm{e}^{-x} + C}$$

5. $\int x^2 \ln(x) \mathrm{d}x$.

Answer. `1 integration by parts` $\int x^2 \ln(x) \mathrm{d}x = \left[\frac{x^3}{3} \ln(x) \right] - \int \frac{x^3}{3} \frac{1}{x} \mathrm{d}x$

$$\boxed{\int x^2 \ln x \mathrm{d}x = \frac{x^3}{3} \ln x - \frac{x^3}{9}}$$

6. $\int_1^2 \ln(u) \mathrm{d}u$.

Answer. `1 integration by parts`

$$\boxed{\int_1^2 \ln u \mathrm{d}u = 2\ln(2) - 1}$$

7. $\int_0^{\pi/2} (x+2) \cos x \mathrm{d}x$.

Answer. `1 integration by parts`

$$\boxed{\frac{\pi}{2} + 1}$$

8. $\int_0^1 (x^2 - x) \, \mathrm{e}^{2x} \, \mathrm{d}x$.

Answer. `2 integrations by parts`

$$\boxed{e - 3}$$

9. $\int_0^1 x \, \mathrm{e}^{-x^2} \, \mathrm{d}x$.

Answer. `substitution` $u = -x^2$`!`

$$\boxed{\frac{1 - \mathrm{e}}{2}}$$

10. $\int_0^{\pi/4} \tan x \mathrm{d}x$.

Answer. `Substitution` $u = \cos x$

$$\boxed{\ln(\sqrt{2})}$$

2 Section 8.2

Exercise 2. Evaluate

1. $\int \sin^2 x \cos x - 2\cos^4 x \sin x \mathrm{d}x$.

Answer. `See your notes`

□

2. $\int \sin^2 t \cos^2 t \mathrm{d}t$

Answer. `See keys for recitation 6`

□

3. $\int \sin t \cos t \mathrm{d}t.$

Answer. `both substitutions` $u = \sin t$ `or` $v = \cos t$ `work.`

$$\boxed{\int \cos t \sin t \mathrm{d}t = \frac{\sin^2 t}{2} + C_1 \text{ or } \frac{-\cos^2 t}{2} + C_2}$$

4. $\int \sin^4 t \mathrm{d}t.$

Answer.

$$\begin{aligned}\int \sin^4 t \mathrm{d}t &= \int \frac{(1-\cos(2t))^2}{4} \mathrm{d}t \\ &= \int \frac{1}{4}(1 - 2\cos(2t) + \cos^2(2t))\mathrm{d}t \\ &= \frac{1}{4} \int 1 - 2\cos(2t) + \frac{\cos(4t)+1}{2} \mathrm{d}t\end{aligned}$$

$$\boxed{\frac{3t}{8} - \frac{\sin(2t)}{4} + \frac{\sin 4t}{32} + C}$$

5. $\int \cos(5t)\sin(3t)\mathrm{d}t.$

Answer. `Use the product-to-sum identity`

$$\cos(5t)\sin(3t) = \frac{1}{2}(\sin(-2t) + \sin(8t)) = \frac{1}{2}(-\sin 2t + \sin 8t)$$

$$\boxed{\frac{1}{4}\cos 2t - \frac{1}{16}\cos 8t + C}$$

6. $\int t\cos(5t)\sin(3t)\mathrm{d}t.$

Answer.

$$\int t \cos 5t \sin 3t \mathrm{d}t = \frac{1}{2}\int t(-\sin(2t) + \sin 8t)\mathrm{d}t$$

$$\boxed{\int t \cos 5t \sin 3t \mathrm{d}t = \frac{1}{4}t\cos 2t - \frac{1}{8}\sin 2t - \frac{1}{16}t\cos 8t + \frac{1}{128}\sin 8t + C}$$

7. $\int_0^{\pi/4} \frac{1+\sin x}{\cos^2 x}\mathrm{d}x.$

Answer. $\int_0^{\pi/4} \frac{1+\sin x}{\cos^2 x}\mathrm{d}x = \int_0^{\pi/4} \frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x}\mathrm{d}x$

$$\boxed{\sqrt{2}}$$

3 Section 8.3

1. $\displaystyle\int_0^1 \frac{u}{\sqrt{1-u^2}}\mathrm{d}u.$

Answer. The integral is improper of type 2 at $u = 1$.

Evaluate $\displaystyle\int_0^t \frac{u}{\sqrt{1-u^2}}\mathrm{d}u.$

Substitution $u = \sin\theta, \quad \mathrm{d}u = \cos\theta\mathrm{d}\theta.$

If $u = 0$, $\theta = 0$. If $u = t$, $\theta = \text{Arcsin}\, t$.

$$\int_0^t \frac{u}{\sqrt{1-u^2}}\mathrm{d}u = \int_0^{\text{Arcsin}\, t} \frac{\sin\theta\cos\theta}{\cos\theta}\mathrm{d}\theta = -\cos(\text{Arcsin}\, t) + 1.$$

If t goes to 1, $\text{Arcsin}\, t \to \frac{\pi}{2}$, $\cos(\text{Arcsin}\, t) \to 0$.

The integral converges and equals 1.

$$\boxed{1}$$

2. $\displaystyle\int_0^1 \sqrt{1-x^2}\mathrm{d}x.$

Answer. Substitution $x = \sin t$

$$\boxed{\frac{\pi}{4}}$$

3. $\displaystyle\int_0^4 \frac{1}{(16+t^2)^2}\mathrm{d}t.$

Answer. See your notes

□

4. $\displaystyle\int \frac{x^4}{\sqrt{1+x^2}}\mathrm{d}x$. Consider $\displaystyle\int \frac{x^3}{\sqrt{1+x^2}}\mathrm{d}x$, or $\displaystyle\int \frac{1}{\sqrt{1+x^2}}\mathrm{d}x$ instead.

Answer. Substitution $x = \tan u$. $\quad (\sec u = \sqrt{1+\tan^2 u} = \sqrt{1+x^2})$

$$\begin{aligned}\int \frac{x^3}{\sqrt{1+x^2}}\mathrm{d}x &= \int \tan^3 u \sec u \mathrm{d}u \\ &= \int (\sec^2 u - 1)(\sec u \tan u)\mathrm{d}u \\ &= \frac{\sec^3 u}{3} - \sec u + C\end{aligned}$$

$$\int \frac{1}{\sqrt{1+x^2}}\mathrm{d}x = \int \sec u \mathrm{d}u = \ln(\sec u + \tan u) + C$$

$$\boxed{\int \frac{x^3}{\sqrt{1+x^2}}\mathrm{d}x = \frac{\sqrt{1+x^2}^3}{3} - \sqrt{1+x^2} + C \qquad \int \frac{1}{\sqrt{1+x^2}}\mathrm{d}x = \ln(\sqrt{1+x^2} + x) + C}$$

4 Sections 8.4

Exercise 3. Find the partial fraction decomposition and evaluate the integral of

1. $f(x) = \dfrac{3x^2+12x+11}{(x+1)(x+2)(x+3)}$.

   Answer.

   $$f(x) = \frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3}, \quad \int f(x)\mathrm{d}x = \ln|x+1| + \ln|x+2| + \ln|x+3| + C$$

2. $g(x) = \dfrac{x}{(x-1)(x^2+1)}$.

   Answer.

   $$g(x) = \frac{1}{2(x-1)} + \frac{-x+1}{2(x^2+1)}, \quad \int g(x)\mathrm{d}x = \frac{1}{2}\ln|x-1| - \frac{1}{4}\ln(x^2+1) + \frac{1}{2}\text{Arctan}\, x + C$$

3. $h(x) = \dfrac{x^4}{(x-1)(x+1)}$.

   Answer. `Do not forget the long division!`

   $$h(x) = x^2 + 1 + \frac{1}{2(x-1)} - \frac{1}{2(x+1)}, \quad \int h(x)\mathrm{d}x = \frac{x^3}{3} + x + \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C$$

4. $j(x) = \dfrac{x^2-3x-2}{(x^2-1)(x-1)}$.

   Answer. `Factorize the denominator`

   $$j(x) = \frac{1}{2(x+1)} + \frac{1}{2(x-1)} - \frac{2}{(x-1)^2}, \quad \int j(x)\mathrm{d}x = \frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + \frac{2}{x-1} + C$$

5. $k(x) = \dfrac{3-2x}{x^2-3x+2}$.

   Answer.

   $$k(x) = \frac{-1}{x-1} - \frac{1}{x-2}, \quad \int k(x)\mathrm{d}x = -\ln|x-1| - \ln|x-2| + C$$

6. $l(x) = \dfrac{x^4}{(x-1)^2(x^2+4)}$.

   Answer. `Don't forget the long division!`

   $$l(x) = 1 + \frac{18}{25(x-1)} + \frac{1}{5(x-1)^2} + \frac{16(2x-3)}{25(x^2+4)}$$

   $$\int l(x)\mathrm{d}x = x + \frac{18\ln|x-1|}{25} - \frac{1}{5(x-1)} + \frac{16\ln(x^2+4)}{25} - \frac{24}{25}\text{Arctan}\left(\frac{x}{2}\right) + C$$

5 Section 8.9

Exercise 4. Evaluate the following integrals of show that they are divergent.

1. $\displaystyle\int_2^6 \frac{x}{\sqrt{x-2}}\mathrm{d}x.$

   Answer. Improper integral of type 2 with singularity at $x = 2$.

$$\int_2^6 \frac{x}{\sqrt{x-2}}\mathrm{d}x = \lim_{t\to 2^+}\int_t^6 \frac{x}{\sqrt{x-2}}\mathrm{d}x$$

   .

   Substitution $u = x - 2$.

$$\int_t^6 \frac{x}{\sqrt{x-2}}\mathrm{d}x = \int_{t-2}^4 \sqrt{u} + \frac{2}{\sqrt{u}}\mathrm{d}u = \frac{40}{3} - \frac{2(t-2)^{3/2}}{3} + 4\sqrt{t-2}$$

   If $t \to 2$, $t - 2 \to 0$, $\displaystyle\int_t^6 \frac{x}{\sqrt{x-2}}\mathrm{d}x \to \frac{40}{3}$

   The integral converges and equals $\dfrac{40}{3}$

2. $\displaystyle\int_0^\infty \frac{1}{(x-3)^2}\mathrm{d}x$

   Answer. Improper of type 1 at ∞, and improper of type 2 at $x = 3$.

   The integral converges if $\displaystyle\int_0^3 \frac{1}{(x-3)^2}\mathrm{d}x$ converges , **and** $\displaystyle\int_3^4 \frac{1}{(x-3)^2}\mathrm{d}x$ converges, **and** $\displaystyle\int_4^\infty \frac{1}{(x-3)^2}\mathrm{d}x$ converges.

   If one (or more) of the 3 integral diverges, $\displaystyle\int_0^\infty \frac{1}{(x-3)^2}\mathrm{d}x$ diverges.

   Let's look at $\displaystyle\int_0^3 \frac{1}{(x-3)^2}\mathrm{d}x.$

$$\int_0^3 \frac{1}{(x-3)^2}\mathrm{d}x = \lim_{t\to 3^-}\int_0^t \frac{1}{(x-3)^2}\mathrm{d}x = \lim_{t\to 3^-}\frac{-1}{t-3} - \frac{1}{3} = \infty$$

   The integral $\displaystyle\int_0^3 \frac{1}{(x-3)^2}\mathrm{d}x$ is divergent therefore

   $\displaystyle\int_0^\infty \frac{1}{(x-3)^2}\mathrm{d}x$ is divergent.

3. $\displaystyle\int_1^e \frac{1}{t\sqrt{\ln t}}\mathrm{d}t.$

   Answer. Improper integral of type 2 with singularity at $t = 1$.

   Using the substitution $u = \ln t$,

   $\displaystyle\int_1^e \frac{1}{t\sqrt{\ln t}}\mathrm{d}t$ is convergent, equals 2

4. $\displaystyle\int_1^2 \frac{\sqrt{x^2-1}}{x}\mathrm{d}x.$

Answer. `Regular integral, not an improper integral!`

`Substitution` $x = \sec u$.

$$\int_1^2 \frac{\sqrt{x^2-1}}{x}\mathrm{d}x = \int_0^{\pi/3} \frac{\tan^2 u \sec u}{\sec u}\mathrm{d}u = \int_0^{\pi/3} \sec^2 u - 1\mathrm{d}u$$

$$\boxed{\sqrt{3} - \frac{\pi}{3}}$$

5. $\displaystyle\int_0^\infty \frac{1}{(x+1)^2(x+2)}\mathrm{d}x.$

Answer. `Improper of type 1 at` ∞.

$$\frac{1}{(x+1)^2(x+2)} = \frac{1}{x+2} - \frac{1}{x+1} + \frac{1}{(x+1)^2}$$

$$\begin{aligned}\int_0^t \frac{1}{(x+1)^2(x+2)}\mathrm{d}x &= \ln(t+2) - \ln(t+1) - \ln(2) - \frac{1}{(t+1)} + 1 \\ &= \ln\left(\frac{t+2}{t+1}\right) - \frac{1}{(t+1)} + 1\end{aligned}$$

$$\boxed{\text{The integral is convergent and equals } 1 - \ln(2)}$$

6. $\displaystyle\int_1^\infty \frac{\operatorname{Arctan} x}{x^2}\mathrm{d}x.$

Answer. `By integration by parts`

$$\begin{aligned}\int_1^t \frac{\operatorname{Arctan} x}{x^2}\mathrm{d}x &= \left[\frac{-\operatorname{Arctan} x}{x}\right]_1^t + \int_1^t \frac{1}{x(x^2+1)}\mathrm{d}x \\ &= \frac{-\operatorname{Arctan} t}{t} + \frac{\pi}{4} + \int \frac{1}{x} - \frac{x}{x^2+1}\mathrm{d}x \\ &= \frac{-\operatorname{Arctan} t}{t} + \frac{\pi}{4} + \ln\left(\frac{x}{\sqrt{x^2+1}}\right) - \ln\left(\frac{1}{\sqrt{2}}\right)\end{aligned}$$

$$\boxed{\text{The integral is convergent and equals } \frac{\pi}{4} + \ln(\sqrt{2})}$$

Exercise 5. Use the comparison theorem to prove that

1. $\displaystyle\int_2^\infty \frac{x^2-2}{x^5+3}\mathrm{d}x$ is convergent.

Answer. `The function` $\dfrac{x^2}{x^5} = x^{-3}$ `whose integral is convergent and is greater than` $\dfrac{x^2-2}{x^5+3}$.

`You have to prove that for all` $x \in [2, \infty)$, $\dfrac{x^2-2}{x^5+3} \leqslant x^{-3}$ `and prove that` $\int_2^\infty x^{-3}\mathrm{d}x$ `is convergent.`

□

2. $\int_1^{\infty} \frac{\sin^2 x}{x^3+5x} \mathrm{d}x$ is convergent.

Answer. $\sin^2 x \leqslant 1$, $x^3+5x \geqslant x^3$.

$0 \leqslant \frac{sin^2 x}{x^3+5x} \leqslant \frac{1}{x^3}$ and $\int_1^{\infty} \frac{1}{x^3} \mathrm{d}x$ is convergent.

By the comparison Theorem, $\int_1^{\infty} \frac{\sin^2 x}{x^3+5x} \mathrm{d}x$ is convergent.

□

3. $\int_0^1 x \cos^4\left(\frac{1}{x}\right) \mathrm{d}x$ is convergent.

Answer. $x \cos^4\left(\frac{1}{x}\right) \leqslant x$.

□

4. $\int_1^{\infty} \frac{\ln(x+1)}{\sqrt{x}} \mathrm{d}x$ is divergent.

Answer. $\ln(x+1) \geqslant \ln(2)$.

$\frac{\ln(x+1)}{\sqrt{x}} \geqslant \frac{\ln 2}{\sqrt{x}}$ with $\int_1^{\infty} \frac{\ln(2)}{\sqrt{x}} \mathrm{d}x$ divergent.

□

5. $\int_0^1 \frac{x^2+3}{1-x} \mathrm{d}x$ is divergent.

Answer. Problem at $x=1$.

$x^2+3 \geqslant 3$.

$\frac{x^2+3}{1-x} \geqslant \frac{3}{(1-x)}$ with $\int_0^1 \frac{3}{1-x} \mathrm{d}x$ divergent.

□

6 Section 9.3

Exercise 6. Find the length of the curve

$$x(t) = \cos^3 t, \quad y(t) = \sin^3 t, \quad t \in [0, \pi/2].$$

Answer.

$$x'(t) = -3\cos^2 t \sin t, \quad y'(t) = 3\sin^2 t \cos t$$

.

$$x'^2(t) + y'^2(t) = 9cos^2 t \sin^2 t(\cos^2 t + \sin^2 t) = 9\cos^2 t \sin^2 t$$

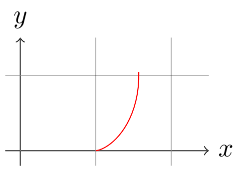
.

$$l = \int_0^{\pi/2} \sqrt{9\cos^2 t \sin^2 t} \mathrm{d}t = \int_0^{\pi/2} 3\cos t \sin t \mathrm{d}t$$

$$\boxed{\frac{3}{2}}$$

Exercise 7. (exam 2 2010) Find the length of the curves

$$x(\theta) = \cos\theta + \theta\sin\theta, \quad y(\theta) = \sin\theta - \theta\cos\theta \quad 0 \leqslant \theta \leqslant \pi/2.$$

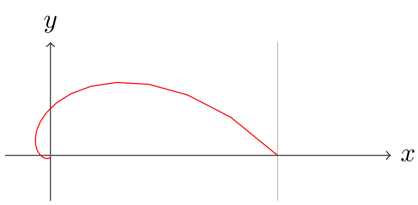


Answer. see `http://www.math.tamu.edu/courses/math152/common-exams/2010c_x2a_sols.pdf`

□

Exercise 8. Find the length of the curve

$$x = \mathrm{e}^{-t}\cos t, \quad y = \mathrm{e}^{-t}\sin t, \quad t \in [0, \infty).$$



Answer.

$$x'(t) = -(\cos t + \sin t)\,\mathrm{e}^{-t}, \quad y'(t) = (-\sin t + \cos t)\,\mathrm{e}^{-t}$$

$$x'^2 + y'^2 = (2\cos^2 t + 2\sin^2 t)\,\mathrm{e}^{-2t} = 2\,\mathrm{e}^{-2t}$$

$$l = \int_0^\infty \sqrt{2}\,\mathrm{e}^{-t}\,\mathrm{d}t$$

.

$$\boxed{l = \sqrt{2}}$$

Exercise 9. Find the length of the curve $x = \ln(\sin y) \quad \pi/6 \leqslant y \leqslant \pi/3$.
Answer.

$$x'(y) = \cot y, \quad 1 + x'^2 = 1 + \cot^2 y = \csc^2 x$$

$$l = \int_{\pi/6}^{\pi/3} \csc x \mathrm{d}x = [\ln(\csc x - \cot x)]_{\pi/6}^{\pi/3}$$

$$\boxed{-\ln(\sqrt{3}) - \ln(2 - \sqrt{3})}$$

7 Section 9.4

Exercise 10. (14p553) Find the area of the surface obtained by rotating the curve

$$y = 1 - x^2, \quad 0 \leqslant x \leqslant 1$$

about the y-axis.
Answer. $y'(x) = -2x, \quad 1 + y'^2 = 1 + 4x^2$

$$\mathcal{A} = 2\pi \int_0^1 x\sqrt{1+4x^2}\mathrm{d}x$$

$$\boxed{\frac{\pi}{6}(5\sqrt{5}-1)}$$

Exercise 11. Given the curve

$$y = \frac{x^3}{6} + \frac{1}{2x} \quad 1 \leqslant x \leqslant 2.$$

1. Find the length of the curve .

   Answer. $y'(x) = \dfrac{x^2}{2} - \dfrac{1}{2x^2} \quad 1 + y'^2 = \dfrac{4x^4 + x^8 - 2x^4 + 1}{4x^4} = \left(\dfrac{x^4+1}{4x^4}\right)^2.$

   $$L = \int_1^2 \frac{x^4+1}{2x^2}\mathrm{d}x$$

   $$\boxed{\frac{17}{12}}$$

2. Find the area of the surface obtained by rotating the curve about the x-axis.
3. Find the area of the surface obtained by rotating the curve about the y-axis.

   Answer.

   $$\mathcal{A} = 2\pi \int_1^2 x\left(\frac{x^4+1}{2x^2}\right)\mathrm{d}x$$

   $$\boxed{\frac{15}{8} + \frac{\ln 2}{2}}$$

Exercise 12. The curve

$$y = x^2, \quad 0 \leqslant x \leqslant 1$$

is rotated

1. about the y-axis. Find the area of the resulting surface.

   Answer. $y'(x) = 2x$, $\sqrt{1+y'^2} = \sqrt{1+4x^2}$.

   $$\begin{aligned}\mathcal{A} &= 2\pi \int_0^1 x\sqrt{1+4x^2}\mathrm{d}x \\ &= 2\pi \int_1^5 \frac{1}{8}\sqrt{u}\mathrm{d}u \qquad \texttt{substitution:} u = 1+4x^2\end{aligned}$$

   $$\boxed{\mathcal{A} = \frac{\pi}{6}(5\sqrt{5}-1)}$$

2. about the y-axis. Find the area of the resulting surface.

Answer.

$$\boxed{\mathcal{A} = 2\pi \int_0^1 x^2 \sqrt{1+4x^2}\mathrm{d}x}$$