Section 6.5:The substitution rule

1 Substitution in indefinite integral

> **Review:** Let f and g 2 differentiable functions. then
> $$(f(g(x)))' = g'(x) \cdot f'(g(x))$$

Exercise 1. Find an antiderivative of

- $\displaystyle\int -2x\,\mathrm{e}^{-x^2}\mathrm{d}x.$

- $\displaystyle\int (2s+3)\sqrt{s^2+3s-4}ds.$

Exercise 2.

- Find $\displaystyle\int \frac{t^2-3t+5}{t}\mathrm{d}t$

- Find $\displaystyle\int \frac{x^2-5x+9}{x-1}\mathrm{d}x.$

> **The substitution rule for indefinite integrals:** If $u = g(x)$ is a differentiable function whose range is an interval I and f a continuous function on I, then
> $$\int f(g(x))g'(x)\mathrm{d}x = \int f(u)\mathrm{d}u.$$

Substitution Method: To evaluate $\int f(x)\mathrm{d}x$

Example: $\displaystyle\int x^2\, \mathrm{e}^{x^3+1}\, \mathrm{d}x$

- Find a substitution $u(x)$ that simplify the integrand

  Example: $u(x) = x^3 + 1$.

- Express $\mathrm{d}u = u'(x)\mathrm{d}x$

  Example: $\mathrm{d}u = 3x^2\mathrm{d}x$.

- Using the expressions of u and $\mathrm{d}u$, eliminate completely the original variable x.

  Example: Replace $x^3 + 1$ by u and $3x^2\mathrm{d}x$ by $\mathrm{d}u$, $\int x^2\, \mathrm{e}^{x^3+1}\, \mathrm{d}x = \dfrac{1}{3}\displaystyle\int \mathrm{e}^u\, \mathrm{d}u$.

- Evaluate the new integral as a function of u, if possible.

  Example: $\dfrac{1}{3}\displaystyle\int \mathrm{e}^u\, \mathrm{d}u = \dfrac{1}{3}\, \mathrm{e}^u + C$.

- Express the solution in term of the original variable x.

  Example: The antiderivatives are $\dfrac{1}{3}\, \mathrm{e}^{u(x)} + C = \dfrac{1}{3}\, \mathrm{e}^{x^3+1} + C$ where C is a constant real number.

Exercise 3. Use the substitution $u = 2x - 3$ to find $\int 4x\sqrt{2x-3}\mathrm{d}x$.

- $\mathrm{d}u =$ $\qquad\qquad$ $\mathrm{d}x$.

- Eliminate x in $\int 4x\sqrt{2x-3}\mathrm{d}x =$

- Find an antiderivative as a function of u:

- Find an antiderivative as a function of x:

Exercise 4. Using a substitution, find $\int t\sqrt{t^2-1}\mathrm{d}t$.

Exercise 5. Find the following indefinite integrals

- $\displaystyle\int \frac{1}{3x+2}\mathrm{d}x$.

- $\int x^2 \, e^{5x^3} \, dx$.

- $\int (x^2 - 1)^3 x^3 dx$.

- $\int \frac{(\ln x)^5}{x} dx$.

- $\int \frac{t^2 - 2t}{t^3 - 3t^2} dt$

2 substitution in definite integral

The substitution rule for definite integrals: If $g'(x)$ is continuous on $[a, b]$ and f is a continuous function on the range of g, then

$$\int_a^b f(g(x)) dx = \int_{g(a)}^{g(b)} f(u) du.$$

Substitution for definite integrals: To evaluate $\int_a^b f(x)\mathrm{d}x$

Example: $\displaystyle\int_0^1 \frac{x^2+2x}{x^3+3x^2+5}\mathrm{d}x.$

- Find a substitution $u(x)$ that simplify the integrand

  Example: $u(x) = x^3 + 3x^2 + 5.$

- Express $\mathrm{d}u = u'(x)\mathrm{d}x$

  Example: $\mathrm{d}u = 3x^2 + 6x\mathrm{d}x.$

- Find the new limit of integration: if $x = a$, $u(x) = ?$ (new lower limit), if $x = b$, $u(x) =?$ (new upper limit).

  Example: If $x = 0$ then $u = 5$. If $x = 1$, $u = 9$.

  The new lower limit is 5 and the new upper limit is 9.

- Using the expressions of u and $\mathrm{d}u$, eliminate completely the original variable x and change the limits of integration.

  Example: Replace x^3+3x^2+5 by u and $3x^2+6x\mathrm{d}x$ by $\mathrm{d}u$, the boundaries by 5 and 9.

  $$\int_0^1 \frac{x^2+2x}{x^3+3x^2+5}\mathrm{d}x = \frac{1}{3}\int_5^9 \frac{1}{u}\mathrm{d}u.$$

- Evaluate the new definite integral if possible.

  Example: $\displaystyle\frac{1}{3}\int_5^9 \frac{1}{u}\mathrm{d}u = \frac{1}{3}[\ln u]_5^9 = \frac{1}{3}(\ln(9) - \ln(5)).$

Exercise 6. Find $\displaystyle\int_0^5 x\sqrt{5-x}\mathrm{d}x$

Exercise 7. Find $\displaystyle\int_e^{e^2} \frac{(\ln x)^5}{x}\mathrm{d}x$

Exercise 8. Find $\displaystyle\int_0^1 x^3\, \mathrm{e}^{x^4}\, \mathrm{d}x$

Exercise 9. Find $\displaystyle\int_0^{\ln(2)} \frac{3\,\mathrm{e}^x}{2+e^x}\mathrm{d}x.$

Exercise 10. Find $\displaystyle\int_0^{\pi/2} \cos x \sin^5 x\mathrm{d}x.$