Section 8-1: Integration by parts

Formula for integration by parts :

$$\int uv'\mathrm{d}x = uv - \int u'v\mathrm{d}x.$$

$$\int_a^b uv'\mathrm{d}x = [uv]_a^b - \int_a^b u'v\mathrm{d}x.$$

Remark:

Exercise 1. Evaluate the integrals

1. $\displaystyle\int_0^3 x\,\mathrm{e}^x\,\mathrm{d}x.$

   Answer. `Integrate by parts:`

$$u(x) = x, \quad u'(x) = 1$$
$$v'(x) = \mathrm{e}^x, \quad v(x) = \mathrm{e}^x$$

$$\int_0^3 x\,\mathrm{e}^x\,\mathrm{d}x = [x\,\mathrm{e}^x]_0^3 - \int_0^3 \mathrm{e}^3\,\mathrm{d}x = 3\,\mathrm{e}^3 - [\mathrm{e}^x]_0^3$$

$$\boxed{\int_0^3 x\,\mathrm{e}^x\,\mathrm{d}x = 2\,\mathrm{e}^3 + 1}$$

2. $\displaystyle\int (x^2 + 3x - 4)\,\mathrm{e}^{2x}\,\mathrm{d}x.$

   Answer. `Integrate by parts:`

$$u(x) = x^2 + 3x - 4, \quad u'(x) = 2x + 3$$
$$v'(x) = \mathrm{e}^{2x}, \quad v(x) = \frac{\mathrm{e}^{2x}}{2}$$

$$\int (x^2+3x-4)\,\mathrm{e}^{2x}\,\mathrm{d}x = (x^2+3x-4)\frac{\mathrm{e}^{2x}}{2} - \int (2x+3)\frac{\mathrm{e}^{2x}}{2}\mathrm{d}x$$

`Second integration by parts:`

$$u(x) = 2x+3, \quad u'(x) = 2$$
$$v'(x) = \mathrm{e}^{2x}, \quad v(x) = \frac{\mathrm{e}^{2x}}{2}$$

$$\int (x^2+3x-4)\,\mathrm{e}^{2x}\,\mathrm{d}x = \frac{1}{2}(x^2+3x-4)\,\mathrm{e}^{2x} - \frac{1}{4}(2x+3)\,\mathrm{e}^{2x} + \frac{1}{2}\int \mathrm{e}^{2x}\,\mathrm{d}x$$

$$\boxed{\int (x^2+3x-4)\,\mathrm{e}^{2x}\,\mathrm{d}x = \left(\frac{x^2}{2}+x-\frac{5}{2}\right)\mathrm{e}^{2x} + C}$$

3. $\displaystyle\int x^3\,\mathrm{e}^{x^2}\,\mathrm{d}x.$

   Answer. `Substitution` $t = x^2$, $\mathrm{d}t = 2x\mathrm{d}x$,

$$\int x^3\,\mathrm{e}^{x^2}\,\mathrm{d}x = \int t\,\mathrm{e}^{t}\,\frac{\mathrm{d}t}{2}$$

   `Integrate by parts:`

$$u(t) = t, \quad u'(t) = 1$$
$$v'(t) = \mathrm{e}^{t}, \quad v(t) = \mathrm{e}^{t}$$

$$\int x^3\,\mathrm{e}^{x^2}\,\mathrm{d}x = \frac{1}{2}(t\,\mathrm{e}^{t}) - \frac{1}{2}\int \mathrm{e}^{t}\,\mathrm{d}t = \frac{1}{2}(t\,\mathrm{e}^{t}) - \frac{1}{2}\,\mathrm{e}^{t} + C$$

$$\boxed{\int x^3\,\mathrm{e}^{x^2}\,\mathrm{d}x = \frac{x^2}{2}\,\mathrm{e}^{x^2} - \frac{1}{2}\,\mathrm{e}^{x^2} + C}$$

Remark:

Exercise 2. Evaluate the integrals

1. $\displaystyle\int x\sin 3x\mathrm{d}x.$

   Answer. `Integrate by parts:`

$$u(x) = x, \quad u'(x) = 1$$
$$v'(x) = \sin(3x), \quad v(t) = -\frac{\cos 3x}{3}$$

$$\int x\sin 3x\mathrm{d}x = -\frac{x\cos 3x}{3} + \int \frac{\cos 3x}{3}\mathrm{d}x$$

$$\boxed{\int x\sin 3x\mathrm{d}x = -\frac{x\cos 3x}{3} + \frac{\sin 3x}{9} + C}$$

2. $\displaystyle\int_0^{\pi/6} (x^2-1)\cos(2x)\mathrm{d}x.$

Answer. `Integrate by parts:`

$$u(x) = x^2 - 1, \quad u'(x) = 2x$$
$$v'(x) = \cos(2x), \quad v(t) = \frac{\sin 2x}{2}$$

$$\int_0^{\pi/6} (x^2-1)\cos(2x)\mathrm{d}x = \left[(x^2-1)\frac{\sin 2x}{2}\right]_0^{\pi/6} - \int_0^{\pi/6} 2x\frac{\sin 2x}{2}\mathrm{d}x$$

`Integrate by parts:`

$$u(x) = x, \quad u'(x) = 1$$
$$v'(x) = \sin(2x), \quad v(t) = -\frac{\cos 2x}{2}$$

$$\int_0^{\pi/6} (x^2-1)\cos(2x)\mathrm{d}x = \left(\frac{\pi^2}{36}-1\right)\frac{\sqrt{3}}{4} - \left[-x\frac{\cos 2x}{2}\right]_0^{\pi/6} - \int_0^{\pi/6} \frac{\cos 2x}{2}\mathrm{d}x$$

$$\boxed{\int_0^{\pi/6} (x^2-1)\cos(2x)\mathrm{d}x = \left(\tfrac{\pi^2}{36}-1\right)\tfrac{\sqrt{3}}{4} + \tfrac{\pi}{24} - \tfrac{\sqrt{3}}{8}}$$

Remark:

Exercise 3. Evaluate the integrals

1. $\displaystyle\int_1^4 x^5 \ln x\mathrm{d}x.$

   Answer. `Integrate by parts:`

$$u(x) = \ln x, \quad u'(x) = \frac{1}{x}$$
$$v'(x) = x^5, \quad v(t) = \frac{x^6}{6}$$
$$\int_1^4 x^5 \ln x\mathrm{d}x = \left[\frac{x^6}{6}\ln x\right]_1^4 - \int_1^4 \frac{x^6}{6x}\mathrm{d}x$$
$$\boxed{\int_1^4 x^5 \ln x\mathrm{d}x = \tfrac{4^6}{6}\ln 4 - \tfrac{4^6}{36} + \tfrac{1}{36}}$$

2. $\displaystyle\int (x+1))\ln(\sqrt{x})\mathrm{d}x.$

3. $\int \ln(x)\mathrm{d}x$

4. $\int (\ln(x))^2\mathrm{d}x$

Remark:

Exercise 4. Evaluate the integrals

1. $\int_0^1 \operatorname{Arctan} x \mathrm{d}x$.

2. $\int \operatorname{Arcsin} 2x \mathrm{d}x$.

3. $\int x \operatorname{Arctan} x.$

Remark:

Exercise 5. Evaluate the integrals

1. $\int_0^1 e^{2x} \sin 3x \mathrm{d}x.$

2. $\int \sin(\ln(x)) \mathrm{d}x.$