Section 8.9: Improper Integrals

Definition of an improper integral of type 1: We write

$$\int_a^\infty f(x)\mathrm{d}x = \lim_{t\to\infty}\int_a^t f(x)\mathrm{d}x$$

$$\int_{-\infty}^b f(x)\mathrm{d}x = \lim_{t\to-\infty}\int_t^b f(x)\mathrm{d}x$$

$$\int_{-\infty}^\infty f(x)\mathrm{d}x = \int_{-\infty}^a f(x)\mathrm{d}x + \int_a^\infty f(x)\mathrm{d}x.$$

If the limits exist and are finite, the integrals are convergent.
Otherwise, they are divergent.

Exercise 1. Determine whether the integrals are convergent or divergent. Evaluate those that are convergent.

1. (3p516) $\displaystyle\int_2^\infty \frac{1}{\sqrt{x+3}}\mathrm{d}x.$

2. (13p517) $\displaystyle\int_0^\infty \frac{1}{(x+2)(x+3)}\mathrm{d}x.$

3. $\displaystyle\int_0^\infty \mathrm{e}^{-2x}\,\mathrm{d}x$ and $\displaystyle\int_{-\infty}^0 \mathrm{e}^{-2x}\,\mathrm{d}x.$

4. $\displaystyle\int_1^\infty \frac{1}{3x+1}\mathrm{d}x$ and $\displaystyle\int_1^\infty \frac{1}{(3x+1)^2}\mathrm{d}x$.

5. $\displaystyle\int_{-\infty}^\infty \frac{1}{1+x^2}\mathrm{d}x$.

6. $\displaystyle\int_{-\infty}^\infty x\,\mathrm{e}^{-x^2}\,\mathrm{d}x$.

Definition of an improper integral of type 2: If f is continuous on $[a,b)$ and is discontinuous at b, then

$$\int_a^b f(x)\mathrm{d}x = \lim_{t\to b^-}\int_a^t f(x)\mathrm{d}x.$$

If f is continuous on $(a,b]$ and is discontinuous at a, then

$$\int_a^b f(x)\mathrm{d}x = \lim_{t\to a^+}\int_t^b f(x)\mathrm{d}x.$$

If the limits exist and are finite, the integrals are convergent.

Exercise 2. Determine whether the following integrals converges or diverges. Evaluate those that are convergent.

1. $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\mathrm{d}x.$

2. $\displaystyle\int_0^1 \frac{1}{x}\mathrm{d}x.$

3. $\displaystyle\int_0^1 \frac{1}{x^2}\mathrm{d}x.$

4. (42p517) $\displaystyle\int_0^1 \frac{\ln(x)}{\sqrt{x}}\mathrm{d}x.$

5. (36p517) $\displaystyle\int_0^{\pi/2} \frac{\cos x}{\sqrt{\sin x}}\mathrm{d}x.$

Comparison Theorem: Let f and g be two non negative functions with $f(x) \geqslant g(x) \geqslant 0$.

- If $\displaystyle\int_a^\infty f(x)\mathrm{d}x$ converges, then $\displaystyle\int_a^\infty g(x)\mathrm{d}x$ converges.
- If $\displaystyle\int_a^\infty g(x)\mathrm{d}x$ diverges, then $\displaystyle\int_a^\infty f(x)\mathrm{d}x$ diverges.

Remark:

Exercise 3. Determine whether the following integrals converge or diverge:

1. $\displaystyle\int_1^{\infty} \frac{\cos^2 x}{x^2} \mathrm{d}x.$

2. $\displaystyle\int_1^{\infty} \frac{1}{\sqrt{x^3+1}} \mathrm{d}x.$

3. $\displaystyle\int_{-\infty}^{\infty} \mathrm{e}^{-x^2}\, \mathrm{d}x.$

4. $\displaystyle\int_0^{\pi/4} \frac{1}{x \cos x} \mathrm{d}x.$