Section 9.3: Arc length

Theorem: Let $\mathcal{C}$ be a curve parametrized by $x(t) = f(t)$, and $y(t) = g(t)$ for $t \in [a, b]$. The length of $\mathcal{C}$ is

$$\mathcal{L} = \int_a^b \sqrt{\left(\frac{\mathrm{d}f}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}g}{\mathrm{d}t}\right)^2}\mathrm{d}t.$$

In particular, if $\mathcal{C}$ is the graph of $y = g(x)$ $(f(x) = x)$,

$$\mathcal{L} = \int_a^b \sqrt{1 + \left(\frac{\mathrm{d}g}{\mathrm{d}x}\right)^2}\mathrm{d}x.$$

Or, if $\mathcal{C}$ is the graph of $x = f(y)$ $(g(y) = y)$,

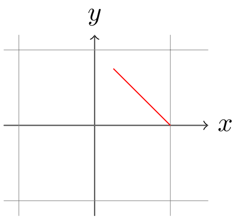
$$\mathcal{L} = \int_a^b \sqrt{\left(\frac{\mathrm{d}f}{\mathrm{d}y}\right)^2 + 1}\mathrm{d}y.$$

Exercise 1. Find the exact length of the curve

$$x = 3t - t^3, \quad y = 3t^2, \quad 0 \leqslant t \leqslant 2.$$

Exercise 2. (exam 2 2011) Find the exact length of the curve parametrized by

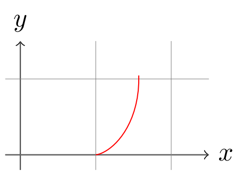
$$x = \sin^2 t, \quad y = \cos^2 t, \quad t \in [0, \pi/3].$$



Exercise 3. Find the exact length of the curve $y = \frac{x^2}{2}, \quad x \in [-1, 1]$.

Exercise 4. (exam 2 2010) Find the length of the curves

$$x(\theta) = \cos\theta + \theta\sin\theta, \quad y(\theta) = \sin\theta - \theta\cos\theta \quad 0 \leqslant \theta \leqslant \pi/2.$$



Exercise 5. Find the length of the curve

$$x = \mathrm{e}^{-t}\cos t, \quad y = \mathrm{e}^{-t}\sin t, \quad t \in [0, \infty).$$