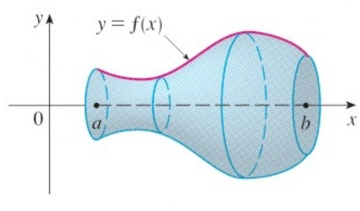
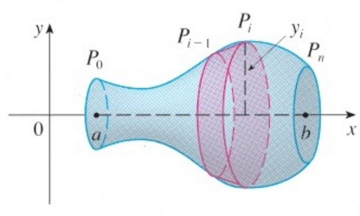
Section 9.4: Area of surface of revolution



(a) Surface of revolution



(b) Approximating band

$$Area = \int_a^b 2\pi r \mathrm{d}s$$

where ds is the variation of the length.

Theorem: Let $\mathcal{C}$ be a curve and $\mathcal{S}$ be the solid obtained by the rotation of $\mathcal{C}$ about the x-axis.

- If $\mathcal{C}$ is parametrized by $x = x(t)$, and $y = y(t)$ for $t \in [a, b]$, then the area of $\mathcal{S}$ is

$$\mathcal{A} = \int_a^b 2\pi y(t)\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t.$$

- If $\mathcal{C}$ is the graph of $y = y(x)$, $x \in [a, b]$,

$$\mathcal{A} = \int_a^b 2\pi y(x)\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x.$$

- If $\mathcal{C}$ is the graph of $x = x(y)$, $y \in [a, b]$,

$$\mathcal{A} = \int_a^b 2\pi y\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2 + 1}\mathrm{d}y.$$

Theorem: Let $\mathcal{C}$ be a curve and $\mathcal{S}$ be the solid obtained by the rotation of $\mathcal{C}$ about the y-axis.

- If $\mathcal{C}$ is parametrized by $x = x(t)$, and $y = y(t)$ for $t \in [a, b]$, then the area of $\mathcal{S}$ is

$$\mathcal{A} = \int_a^b 2\pi x(t) \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2} \mathrm{d}t.$$

- If $\mathcal{C}$ is the graph of $y = y(x)$, $x \in [a, b]$,

$$\mathcal{A} = \int_a^b 2\pi x \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \mathrm{d}x.$$

- If $\mathcal{C}$ is the graph of $x = x(y)$, $y \in [a, b]$,

$$\mathcal{A} = \int_a^b 2\pi x(y) \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2 + 1} \mathrm{d}y.$$

Exercise 1. Find the area of the surface obtained by rotating the curve

$$y = \cos x, \quad x \in [0, \pi]$$

about the x-axis.

Exercise 2. (2p553) Find the area of the surface obtained by rotating the curve

$$y^2 = 4x + 4, \quad 0 \leqslant x \leqslant 8$$

about the x axis.

Exercise 3. (28p553) Find the surface area generated by rotating the curve

$$x(t) = \mathrm{e}^t - t, \quad y(x) = 4\,\mathrm{e}^{t/2}, \quad 0 \leqslant t \leqslant 1$$

about the y-axis.

Exercise 4. (12p553) The curve

$$x = \sqrt{2y - y^2} \quad 0 \leqslant y \leqslant 1$$

is rotated about the y-axis. Find the area of the resulting surface.

Exercise 5. (23p553) Find the area of the surface obtained by rotating the curve

$$x = t^3, \quad y = t^2, \quad 0 \leqslant t \leqslant \frac{2}{3}$$

about the x-axis.

Exercise 6. (14p553) Find the area of the surface obtained by rotating the curve

$$y = 1 - x^2, \quad 0 \leqslant x \leqslant 1$$

about the y-axis.