University of Washington

Chapter 1

1 Section 1, Fundamental Concepts

Assume known the notions of

- Sets, subsets, proper subset, element
- inclusion, proper inclusion, union, intersection, difference, complement

Theorem (Morgan's Law: Given 2 sets A and B,

$$(A \cup B)^c = A^c \cap B^c$$

$$(A \cap B)^c = A^c \cup B^c$$

Definition: Given $\mathcal{A}$ a collection of sets,

- the arbitrary union, written $\bigcup_{A \in \mathcal{A}} A$, is the set of elements that belong to at least one set A of $\mathcal{A}$.
- The arbitrary intersection, is written $\bigcap_{A \in \mathcal{A}} A$, is the set of elements that belong to every sets A of $\mathcal{A}$.

Example:

- If $\mathcal{A} = \left\{ \left[\frac{1}{n}, 10 \right], n \in \mathbb{N} \right\}$, then $\bigcup_{A \in \mathcal{A}} A = (0, 10]$ and $\bigcap_{A \in \mathcal{A}} A = [1, 10]$.
- If $\mathcal{A} = \left\{ \left(\frac{1}{x}, x^2 \right), x \geqslant 1 \right\}$, then $\bigcup_{A \in \mathcal{A}} A = (0, \infty)$ and $\bigcap_{A \in \mathcal{A}} A = \{1\}$.

2 Section 2, Functions

Definitions: A function f from A to B is a set f of pairs (a, b) such that any a in A belongs to at most one pairs in f.
We usually use the notation $f(a) = b$ for the pair (a, b).
The set of elements a of A that effectively belong to a pair of f is the domain of f.
The set of elements b in B that effectively belong to at least a pair of f is the range of f.

Definition: Given 2 functions

$$f : A \to B, \qquad g : B \to C,$$

the composite function $f \circ g$ is the set of pairs (a, c) such that $(a, b) \in f$ and $(b, c) \in g$ for some $b \in B$.

definition: Given a function $f : A \to B$

- f is injective if any element b in B belongs to at most one pair of f.
- f is surjective if any element b in B belongs to at least one pair of f.
- f is bijective if f is injective and surjective.

Definition: Let f be a function from A to B, Let A_0 be a subset ao A and B_0 a subset of B.

- The image of A_0 under f is the set $f(A_0) = \{b \in B | b = f(a) \text{ for } a \in A_0\}$.
- The pre image of B_0 under f is the set $f^{-1}(B_0) = \{a \in A | f(a) \in B_0\}$.

Remark: The pre image exist even if f is non injective.

Theorem: Let f be a function for A to B, A_1, A_2 , B_1, B_2 be 2 subsets of A and of B respectively,

- $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.
- $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$.
- $f(A_1 \cap A_2) \subset f(A_1) \cap f(A_2)$.
- $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$.

Example: $f(x) = x^2$, $A_1 = [-3, 1]$, and $A_2 = [-1, 2]$

$$f(A_1 \cap A_2) = [0, 1], \qquad \text{and} \qquad f(A_1) \cap f(A_2) = [0, 4]$$

Lemma: Let f be a function from A to B,
if there exists 2 functions g and h from B to A such that

- for any a in A, $g \circ f(a) = a$
- for any b in B, $f \circ h(b) = b$

then f is bijective and $f^{-1} = h = g$.

3 Section 3, Relations

Definition: A relation $\mathcal{R}$ on a set A is a subset of $A \times A$

Example: If $A = \{1, 2, 3\}$ the relation $<$ corresponds to the subset $\{(1, 2), (1, 3), (2, 3)\}$ usually written $\{1\mathcal{R}2, 1\mathcal{R}3, 2\mathcal{R}3\}$.

3.1 Equivalence relation

Definition: A Relation $\sim$ is an equivalence relation on A if

- For any x in A, $x \sim x$ (reflexivity)
- If $x \sim y$ then $y \sim x$ (symmetry)
- If $x \sim y$ and if $y \sim z$, then $x \sim z$ (transitivity)

Example:

- $x \sim y$ on the integers if x and y have the same remainder by the division by 3.
- $x \sim y$ on $\mathbb{R}$ if $x^2 = y^2$.

Definition: The equivalence class E_x determined by x is the set of all the element in relation with x.

$$E_x = \{y | x \sim y\}.$$

Example: $\mathbb{Q}$ the elements can be viewed as the equivalence classes of fractions that are equal.

Theorem: Given an equivalence relation on a set A, the equivalence classes form a partition of A.

3.2 Order relation

Definition: A relation $\mathcal{R}$ is an order relation if

- For any x, $x\mathcal{R}x$ (reflexivity)
- For any $x \in A$ and any $y \in A$, then $x\mathcal{R}y$ and $y\mathcal{R}x$, then $x = y$ (antisymmetry)
- If $x\mathcal{R}y$ and if $y\mathcal{R}z$, then $x\mathcal{R}z$ (transitivity)

Definiton: A relation $\mathcal{R}$ is a strict order relation if

- For any x, and for any y if $x \neq y$ then $x\mathcal{R}y$ or $y\mathcal{R}x$. (comparability)
- For no $x \in A$, $x\mathcal{R}x$. (nonreflexivity)
- If $x\mathcal{R}y$ and if $y\mathcal{R}z$, then $x\mathcal{R}z$ (transitivity)

Example: $(x_1, y_1) \leqslant (x_2, y_2)$ in $\mathbb{R}^2$ if

- $x_1^2 - y_1 < x_2^2 - y_2$
- or, $x_1^2 - y_1 = x_2^2 - y_2$ and $x_1 \leqslant x_2$.
- Compare (1,3) and (2,0)
- Compare (3,5) and (2,0)
- Prove that $\leqslant$ is an order relation
- What is the interval $[(0,0),(2,1)]$?

Definition: Given an order relation $\leqslant$ on A and $a \leqslant b$ 2 elements of A. The interval $[a, b]$ is the set of all the element x of A such that $a \leqslant x \leqslant b$.

- Let $S = [-1, 1] \times [-1, 1]$. Find the smallest element of S, the greatest element of S.

4 Section 5, Cartesian Products

Definition: Let A_i a finite collection of sets. An element of $A_1 \times A_2 \times \cdots \times A_n$ is a n-tuple in the form $(x_1, x_2, \cdots, x_n)$ where $x_i \in A_i$ for any $i \in \{1 \cdots n\}$.

Definition: Given a set X, X^ω is the set of all sequences of elements of X.

5 Section 6, Finite Sets

Definition: A is a finite set if A is empty or if there is a bijection from A to $\{1, \cdots n\}$.

Theorem: The number n in the previous definition is unique. It is the cardinality of A

Corollary:

- Let A is a finite set. There is no bijection from A to a proper subset of A.
- Z^+ is infinite

Corollary: Let B a non empty set, the 3 statements are equivalent

- B is finite.
- there is a surjection from $\{1, \cdots, n\}$ onto B for some integer n.
- there is an injection from B to $\{1, \cdots, m\}$ for some integer m.

6 Countable sets

Definition: A set A is countable if

- A is finite
- or if there is a bijection from A to $\mathbb{Z}$ (countably infinite)

Theorem: The 3 statements are equivalent

- A is countable
- there is an injection from A to $\mathbb{Z}$
- there is a surjection from $\mathbb{Z}$ onto A

Theorem:

- $\mathbb{Q}$ is countably infinity
- $\mathbb{R}$ in not countable