Chapter 2

1 Section 12, Topological spaces

Definition: A topology on a set X is a collection τ of subsets of X called open sets such that

1. $\varnothing$ and X are in τ.
2. The arbitrary union of elements of τ is in τ.
3. the finite union of elements of τ is in τ.

Example: The standard topology τ of $\mathbb{R}$ is the sets of all the intervals (a, b), their finite intersections and their arbitrary unions.

2 Section 13, Basis for a topology

Definition: A basis $\mathcal{B}$ on a set X is a collection of subsets such that

1. each x in X belongs to at least one element of $\mathcal{B}$,
2. If x belongs to $B_1 \in \mathcal{B}$ and $B_2 \in \mathcal{B}$ then there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.

The topology generated by $\mathcal{B}$ contains all the elements $\mathcal{U}$ such that for all $x \in \mathcal{U}$, there is $B \in \mathcal{B}$ that satisfies $x \in B \subset \mathcal{U}$.

Properties: Given a basis $\mathcal{B}$ and τ the topology generated by $\mathcal{B}$,

1. Every elements of $\mathcal{B}$ are in τ.
2. τ is the collection that contains all the arbitrary unions of elements of B.
3. τ is a topology.

Proposition: X is a topological space with a topology τ. If $\mathcal{C}$ is a collection of open sets such that for any open $\mathcal{U}$ of τ, and for any x in $\mathcal{U}$, there exists C in $\mathcal{C}$ such that $x \in C \subset \mathcal{U}$, then $\mathcal{C}$ is a basis for the topology τ.

3 Section 20, Metric spaces

Definition: A metric on a set X is a function

$$d : X \times X \to \mathbb{R}^+ = [0, \infty)$$

such that

1. $d(x, y) = 0 \implies x = y$.
2. for any x and y, $d(x, y) = d(y, x)$.
3. for any x, y, and z, $d(x, y) \leqslant d(x, z) + d(z, y)$.

Definition: Given a metric on a set X, the metric topology on X is the topology generated with the ϵ-ball $B(x, \epsilon) = \{y | d(x, y) < \epsilon\}$ for any x and any $\epsilon > 0$.

Definition: A topological space X is metrizable if there exists a metric that induce the same topology.

4 Continuous functions

Definition: A function f on a topological space is continuous if the preimage of any open sets is an open set.

Theorem: Given a topological space X with a basis $\mathcal{B}$, f is continuous iff the preimage of any element of $\mathcal{B}$ is an open set.

Theorem: If f is continuous, then f is $\epsilon - \alpha$ continuous.

Definition: Given 2 topological spaces X and Y and f a function from X to Y. f is an homeomorphism if

- f is continuous
- f is bijective
- f^{-1} is continuous

An homeomorphism induce a 1-1 correspondence between the topology of X and the topology of Y.

5 Topology comparison

Definition: (page 77)Given 2 topologies on a set X. If $\tau \subset \tau'$ then

- τ' is finer than τ.
- τ is coarser than τ'

If $\tau \subset \tau'$ and $\tau' \subset \tau$ then the two topologies are comparable.

Theorem: (page 81) Let $\mathcal{B}$ and $\mathcal{B}'$ be 2 basis for 2 topology τ and τ' respectively
τ' is finer than τ iff

$$\forall x \in X, \forall b \in \mathcal{B} | x \in B \implies \exists B' \in \mathcal{B}' | x \in B' \subset B$$

Theorem: If τ and τ' are 2 topologies on the same set X τ and τ' are comparable iff the identity function $id : (X, \tau) \to (X, \tau')$ is an homeomorphism.

6 Subspace Topology

Definition: Let X be a topological space with a topology τ and Y be a subset of X, the subspace topology τ_Y is the set of elements $\mathcal{U} \cap Y$ where $\mathcal{U} \in \tau$.

Remark: It is a topology.

Theorem: If $\mathcal{B}$ is a basis for the topology τ on X, then the set $\{B \cap Y : B \in \mathcal{B}\}$ is a basis for the subspace topology.

Theorem: Let A be a subspace of the topological space X, and Y be a topological space then

- the inclusion $\begin{array}{cccc} j: & A & \to & X \\ & a & \mapsto & a \end{array}$ is continuous.
- If $f : X \to Y$ is continuous, then $f_{/A} A \to Y$ is continuous.

Theorem: If $X = \bigcup_{\alpha \in J} \mathcal{U}_\alpha$ where $\mathcal{U}_\alpha$ is an open set in X, the function $f : X \to Y$ is continuous iff the restriction $f_{/\mathcal{U}_\alpha}$ is continuous for every $\alpha \in J$.

7 Product Topology

7.1 Cartesian product

Definition: Let X and Y be 2 topological spaces, the product topology is generated by the basis $\mathcal{U} \times \mathcal{V}$ where $\mathcal{U}$ is an open set of X and $\mathcal{V}$ is an open set of Y.

Theorem: If the topologies on X and Y are generated by two bases $\mathcal{B}_X$ and $\mathcal{B}_Y$, then the product topology is generated by the basis $\{B_X \times B_Y$ where $B_X\}$ is an element of the basis $\mathcal{B}_X$ and B_Y is an element of $\mathcal{B}_Y$.

Theorem: Given X and Y 2 topological spaces, the functions

- The first projection $\begin{array}{rccc}\pi_1: & X\times Y & \to & X\\ & (x,y) & \mapsto & x\end{array}$ is continuous for the product topology and the topology on X.
- The second projection $\begin{array}{rccc}\pi_2: & X\times Y & \to & Y\\ & (x,y) & \mapsto & y\end{array}$ is continuous for the product topology and the topology on Y.
- If the topology on X is generated by a metric d, then the function $\begin{array}{rccc}d: & X\times X & \to & \mathbb{R}\\ & (x_1,x_2) & \mapsto & d(x_1,x_2)\end{array}$ is continuous for the product topology and the standard topology on $\mathbb{R}$.

Theorem: If A is a subspace of X and B is a subspace of Y, the product topology of the subspaces is identical to the subspace topology of the product.

7.2 Arbitrary product

Definition: Let X_α for $\alpha \in J$ be a collection of topological spaces. 2 topologies may be defined on the product $\prod_{\alpha\in J} X_\alpha$.

- The box topology generated by the basis

$$\left\{\prod_{\alpha\in J}\mathcal{U}_\alpha \text{ for any open } \mathcal{U}_\alpha \subset X_\alpha\right\}$$

- The product topology generated by the basis

$$\left\{\prod_{\alpha\in J}\mathcal{U}_\alpha : \mathcal{U}_\alpha \subset X_\alpha \text{ open and } \mathcal{U}_\alpha = X_\alpha \text{ except for a finite number of } \alpha\right\}$$

Remark: Study the continuity of the function $f : t \mapsto (t,t,t,\cdots,t,\cdots) \in \mathbb{R}^\omega$ for the 2 topologies.

Remark: the box topology is finer than the product topology.

Theorem: A function $f : A \to \prod_{\alpha\in J} X_\alpha$ is continuous for the product topology iff each function $f_\beta = \pi_\beta \circ f$ are continuous.

8 Quotient Topology

Definition: Let X and Y be 2 topological spaces, and p be a surjective map from X to Y,
p is a quotient map iff an set $\mathcal{U}$ is open in Y iff $p^{-1}(\mathcal{U})$ is open in X.

Theorem/Definition: Let X be topological space, and p be a surjective map from X to Y,
There exists a unique topology τ on Y such that p is a quotient map.
τ is the quotient topology.

Theorem: Let $p : X \to Y$ be a quotient map and $g : X \to Z$ be a map that is constant on each $p^{-1}y$ for $y \in Y$, then there exists a map $f : Y \to Z$ such that $f \circ p = g$.
f is continuous iff g is continuous.

9 Closed sets, interior and closure

9.1 Closed sets

Definition: A set A is closed if its complement A^c is open.

Theorem: Let X be a topological space

- $\varnothing$ and X are closed
- Any arbitrary intersection of closed sets is closed
- Any finite union of closed set is closed

Theorem: Let X be a topological space and Y be a subspace of X,
A is closed in Y iff there exists a closed set C in X such that $A \cap Y = C \cap Y$.

Theorem: Let A and B be 2 closed sets respectively for the topological space X and Y, then $A \times B$ is a closed set for the product topology of $X \times Y$

9.2 Interior, Closure

Definition: Let A be a subset of a topological space X,

- The interior of A, $int(A)$ is the union of all the open sets included in A.
- The closure of A, written $\overline{A}$, is the intersection of all the closed set containing A.

Remark: The interior of A is the greatest open set (for the relation inclusion) included in A.
The closure of A is the smallest closed set (for the relation inclusion) containing A.

Theorem: If Y is a subspace of a topological space X, and A be a subset of Y. The closure of A in Y is the intersection of the closure of A in X with Y.

Theorem: Given a subset A of a topological space X, an element x of X is in the closure of A iff any open $\mathcal{U}$ containing x intersects A.
If the topology on X is generated by a basis, an element x of X is in the closure of A iff any element of the basis B containing x intersects A.

9.3 Accumulation point

Definition: Let A be a subset of a topological space X, An element x of X is an accumulation point if any open $\mathcal{U}$ containing x (neighborhood of x), contains at least an element of A that is not x, i. e.

$$\mathcal{U} \cap A \setminus \{x\} \neq \varnothing$$

Theorem: Let A be a subset of a topological space and A' the set of accumulation points of A, Then $\overline{A} = A \cup A'$.