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Chapter 3

1 Connectedness

1.1 Connectedness

Definition: A space X is connected if it cannot be the union of two disjoint non empty open sets.

Theorem: The 3 statements are equivalent

- A space X is connected.
- X cannot be the union of 2 disjoints non empty closed sets.
- The only subsets of X that are both open and closed are X and $\varnothing$.

Corollary: If A is connected in a topological space X and $A \subset \mathcal{U}_1 \cup \mathcal{U}_2$ where $\mathcal{U}_1$ and $\sqcap_2$ are 2 disjoint open sets then A is entirely included in one of the open sets.

Theorem: The union of a collection of connected subspaces that have a point in common is connected.

Theorem: Given a connected subset A of a topological space X, If B satisfies $A \subset B \subset \overline{A}$ then B is connected

Corollary: If A is connected, then the closure $\overline{A}$ is connected.

Theorem: The image of a connected space under a continuous function is connected.

Theorem: A finite Cartesian product of connected spaces is connected.

Definition: Given a set Y and a total order relation on Y, the order topology is generated by the intervals $(x, y) = \{z \in Y : x < z < y\}$

Theorem (intermediate value): Let $f : X \to Y$ a continuous function from a connected space X to a totally ordered space Y. Then for any two point a and b in X, and for any value $r \in Y$ between $f(a)$ and $f(b)$, there exists c in X such that $f(c) = r$.

1.2 Path connected

Definition: A set A is path connected if for any 2 points a and b in A, there exists a path entirely included in A, $c : [0,1] \to A$, such that $c(0) = a$ and $c(1) = b$.

Theorem: If A is path connected, then A is connected.

Example: The set $\overline{G} = \left\{ \left(x, \sin\left(\frac{1}{x}\right) \right) \in \mathbb{R}^2, x \in (0,1] \right\} \cup \{0\} \times [-1,1]$ is connected and not path connected.

2 Compactness

2.1 Compactness

Definition: Let X be a topological space.
A cover of X is a collection $\mathcal{A}$ of subsets of X such that $X = \bigcup_{A \in \mathcal{A}} A$.
A subcover $\mathcal{B}$ of X is a subset of $\mathcal{A}$ that covers X, i.e. $X = \bigcup_{A \in \mathcal{B}} A$.

Definition: A set C in a topological space X is compact if for any cover of C with open sets, there is a finite subcover of C.

Example:

- $[0,1]$ is compact
- If x_n is a convergent sequence and x is its limit, then $S = \{x, x_n, n \in \mathbb{N}\}$ is compact.

Theorem: Every closed set of a compact set is compact.

Theorem: The compact sets of $\mathbb{R}$ are the closed and bounded sets.

Theorem: If X be a compact topological space, and f be a continuous function from X to Y then $f(X)$ is compact.

Theorem: If f is a continuous function from a compact set X to $\mathbb{R}$, then for any a and b in X, there exists c in X such that $f(c)$ is between $f(a)$ and $f(b)$.

Theorem: The product of finitely many compacts is compact.

Remark: The theorem is also true for an arbitrary product. It won't be proved in class.

Definition: If (X,d) is a metric space, A is a non empty subset of X, and x an element of X, the distance between x and A is defined by

$$d(x,A) = \inf(d(x,a), \forall a \in A)$$

Remark:

- $d(x,A)$ is a continuous function of x.
- $d(x,A) = 0$ iff $x \in \overline{A}$

Theorem: Let $\mathcal{A}$ be a cover by open sets of a metric space X. If X is compact, there exists a positive number δ such that any ball with diameter δ is entirely included in an element of $\mathcal{A}$.

Theorem: If f is continuous from (X, d_X) a compact metric space to (Y, d_y) a metric space, then f is uniformly continuous.

2.2 Hausdorff Space

Definition: (page 88) A topological space X is Hausdorff space if for any $x_1 \neq x_2$, there exists 2 disjoint open sets $\mathcal{U}_1$ and $\mathcal{U}_2$ such that $\mathcal{U}_1$ contains x and $\mathcal{U}_2$ contains y.

Theorem: If X is Hausdorff, any sequence has at most one limit point.

Definition: The order topology on X is generated by the open sets

- (a, b) for any a, b in X.
- $[a_0, b)$ for any b in X and a_0 being a smallest element of X
- $(a, b_0]$ for any b in X and b_0 is the greatest element of X

Theorem:

- A totally ordered space is Hausdorff for the order topology.
- The product of 2 Hausdorff space is Hausdorff.
- A subspace of a Hausdorff space is Hausdorff.
- Metric spaces are Hausdorff
- $\mathbb{R}^n$ is Hausdorff.

Theorem: Any compact space of a Hausdorff set is closed.

Theorem: Let $f : X \to Y$ be a continuous function. If X is compact and Y is Hausdorff, then f is an homeomorphism.

Theorem: Let X be a non empty compact Hausdorff space, if X has no isolated point, then X is non countable.