University of Washington

Last Name (PRINT): ______________________
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Section: ______________________

Summer 2015 – Introductory Real Analysis II
First Examination

Instructions

1. The use of all electronic devices and any additional resources is prohibited. Failure to comply may result in terminating your midterm early
2. Present your solutions in the space provided. Show all your work neatly and concisely. You will be graded not merely on the final answer, but also on the quality and correctness of the work leading up to it.
3. Good luck.

Scholastic dishonesty will not be tolerated.
The work on this test is my own.

Signature:______________________________

Grade:

Exercise 1. Find the radius of convergence of

1. $F(x) = \displaystyle\sum_{n=0}^{\infty} \frac{\ln(n)}{2^n n^2} x^{2n}$

2. $G(x) = \displaystyle\sum_{n=0}^{\infty} \frac{x^{n^2}}{3^n}$

Exercise 2. The power series $\sum_{n=0}^{\infty} a_n x^n$ has a radius of convergence $R > 0$.

Determine the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n^2 x^n$.

Justify your answer.

Exercise 3. Given the power series $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2n+1}$.

1. Find the radius of convergence, R.

2. Evaluate the power series $f(x)$.

Exercise 4. Determine whether the following integrals are convergent.

1. $\displaystyle\int_0^\infty \frac{2+\ln(x)}{x+3}\mathrm{d}x$

2. $\displaystyle\int_1^\infty \frac{\cos t}{\sqrt{e^t-1}}\mathrm{d}t.$

3. $\displaystyle\int_0^1 \frac{1}{\sqrt{t^3+4t^2+t}}\mathrm{d}t.$

Exercise 5. Given the function $\phi(x) = \displaystyle\int_1^\infty \frac{1}{1+t^x}\mathrm{d}t$.

1. For which values of x is ϕ convergent?
2. Prove that ϕ is continuous at any x for which the integral is convergent.
3. Determine the values of x where ϕ is differentiable.

1 Power Series

1.1 Radius of convergence

Definition 1. A power series is a series of functions in the form $f(x) = \sum_{k=0}^{\infty} a_n x^n$.

Theorem 2. Given a power series $f(x) = \sum_{n=-}^{\infty} a_n x^n$, if there exists x_0 such that the series $f(x_0)$ is convergent, then for any x such that $|x| < |x_o|$, the series $f(x)$ is absolutely convergent.

Theorem 3. Given a power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$, exactly one of the 3 statements is true

- f is convergent only for $x = 0$. We say that the radius of convergence is 0.
- f is convergent for any x. We say that the radius of convergence is infinite.
- there exists a positive real R, called radius of convergence such that
  - if $|x| < R$, then the series is absolutely convergent.
  - if $|x| > R$, then the series is divergent.

Theorem 4. Let f be a power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$.

If the limit $\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$ exists or is infinite, then the radius of convergence is $R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$.

1.2 Uniform convergence of power series

Theorem 5. Let f be a power series with a positive or infinite radius of convergence R.
For any $0 < r < R$, the power series is uniformly convergent on $[-r, r]$.

Theorem 6. Let $f = \sum_{n=0}^{\infty} a_n x^n$ be a power series with a positive or infinite radius of convergence R.
The series f is continuous on the open interval $(-R, R)$.
Moreover, for any a and b in the open interval $(-R, R)$,

$$\int_a^b f(x)\mathrm{d}x = \sum_{n=0}^{\infty} a_n \left(\frac{b^{n+1} - a^{n+1}}{n+1}\right)$$

1.3 Differentiability of power series

Theorem 7. The power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ have the same radius of convergence.

Theorem 8. Let $f = \sum_{n=0}^{\infty} a_n x^n$ be a power series with positive or infinite radius of convergence R.

Then f is differentiable at any point in open interval $(-R, R)$ and its derivative is $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$.

By induction, f is differentiable infinitely many times on the open disk of convergence.
And, $a_n = \frac{f^{(n)}(0)}{n!}$.

Theorem 9. If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series with a positive radius of convergence, then $a_n = \dfrac{f^{(n)}(0)}{n!}$

Theorem 10. If $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^2$ on a neighborhood of 0, then for any n, $a_n = b_n$.

1.4 Product and quotient of power series

Definition 11. If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ are 2 powers series with 2 positive radius of convergence R_a and R_b, Then the product of the power series is a power series $h(x0 = \sum_{n=0}^{\infty} c_n x^n$ with $c_n = \sum_{k=0}^{n} a_k b_{n-k}$

Its radius of convergence R is greater of equal to $\min(R_a, R_b)$.

Theorem 12. Given 2 power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ with $b_0 \neq 0$, then $\dfrac{\sum a_n x^n}{\sum b_n x_n}$ can be represented by a power series $\sum c_n x^n$.

To find the power series $\sum c_n x^n$, you may use a long division or notice that $\sum a_n x^n = \left(\sum b_n x^n\right)\left(\sum c_n x^n\right)$. The latest method leads to a triangular system of equations

$$\begin{aligned}
a_0 &= b_0 c_0 && \Longrightarrow c_0 = \cdots \\
a_1 &= b_0 c_1 + b_1 c_0 && \Longrightarrow c_1 = \cdots \\
&\vdots && \vdots \\
a_n &= b_0 c_n + b_1 c_{n-1} + \cdots + b_{n-1} c_1 + b_n c_0 && \Longrightarrow c_n = \cdots
\end{aligned}$$

A third method involves the geometric series formula $\sum_{n=0}^{\infty} x^n = \dfrac{1}{1-x}$.

For example, the function $\dfrac{1}{1+x^2}$ can be represented by the power series $\dfrac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$.

1.5 Inferior limit, superior limit

Definition 13. Given a sequence u_n,

$$\overline{\lim_{n\to\infty}} x_n = \lim_{n\to\infty} \left(\sup\{x_k ; k \geqslant n\}\right)$$

$$\underline{\lim} x_n = \lim_{n\to\infty} \left(\inf\{x_k ; k \geqslant n\}\right)$$

Example: If $x_n = (-1)^n \left(1 + \dfrac{1}{n}\right)$, then $\overline{\lim} x_n = 1$ and $\underline{\lim} x_n = -1$

Theorem 14. Given a power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$, then its radius of convergence R satisfy

$$\frac{1}{R} = \overline{\lim_{n\to\infty}} |a_n|^{1/n}$$

Example: The radius of convergence of the series $\sum_{n=0}^{\infty} \dfrac{x^{n^2}}{2^n}$ is 1.

1.6 Analytic function

Definition 15. Let f be a real valued function defined on (a,b) such that f has derivative of all order at any point x in (a,b).

The function f is analytic on (a,b) is for any x_0 in (a,b), there exists a neighborhood of x_0 such that

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

on the neighborhood of x_0.

Example

- $e^x = \sum_{k=0}^{\infty}$ is analytic on $\mathbb{R}$. Using Euler formula, $\cos x$. $\sin x$, $\tan x$ are analytic on $\mathbb{R}$.
- $\dfrac{1}{1-x}$ is analytic on $(-\infty, 1)$, and on $(1,\infty)$.

  Therefore $\operatorname{Arctan} x$ is analytic on $\mathbb{R}$, $\ln(1+x)$ is analytic on $(-1,\infty)$...
- The function $x^{5/2}$ is not analytic at 0 since $x^{5/2}$ does not have derivatives of all orders.
- The function $f(x) = \begin{cases} 0 & x = 0 \\ e^{-1/x^2} & x \neq 0 \end{cases}$ has derivatives of all order at $x = 0$.

  All the successive derivative at $x = 0$ are 0.

  The Taylor series of f, T_f, is 0. On any neighborhood of f, the Taylor series T_f and f are different functions. Therefore f is not analytic at 0.

Theorem 16. Let f be a function defined on (a,b) such that f has derivative of all order on (a,b) and $f^{(n)}(x) \geqslant 0$ at any $x \in (a,b)$.

Then f is analytic on (a,b).

Theorem 17.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n!} \qquad \ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

$$\sin x \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad \operatorname{Arctan} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

2 Improper integrals

2.1 Improper integral of the first kind and of the second kind

Definition 18. Improper of the first kind

Let f be an integrable function over every finite intervals $[a,c]$ for $a < c$.

- The improper integral of the first kind $\displaystyle\int_a^{\infty} f(x)\mathrm{d}x$ is defined by

$$\int_a^{\infty} f((x)\mathrm{d}x = \lim_{c\to\infty} \int_a^c f(x)\mathrm{d}x$$

  if the limit exists.

- If the limit exists, the improper integral is convergent.
- If the limit does not exists, the improper integral is divergent.

Definition 19. Improper of the second kind
Let f be a function on $[a,b)$, integrable over every finite intervals $[a,c]$ for any c, $a<c<b$.

- The improper integral of the second kind $\displaystyle\int_a^b f(x)\mathrm{d}x$ is defined by

$$\int_a^b f((x)\mathrm{d}x = \lim_{c\to\infty}\int_a^c f(x)\mathrm{d}x$$

  if the limit exists.

- If the limit exists, the improper integral is convergent.
- If the limit does not exists, the improper integral is divergent.

Theorem 20. $\displaystyle\int_0^1 t^\alpha \mathrm{d}t$ is convergent for $\alpha > -1$.
$\displaystyle\int_1^\infty t^\alpha \mathrm{d}t$ is convergent for $\alpha < -1$.

2.2 Improper integral with non negative integrand

Theorem 21. Let b be a finite real number or b is infinite.
Let f be a **non negative** function integrable on any interval $[a,c]$ for $a<c<b$.
The improper integral $\int_a^b f(t)\mathrm{d}t$ is convergent iff $F(x)=\int_a^x f(t)\mathrm{d}t$ is bounded for $x\in[a,b)$.

Theorem 22. Let b be a finite number or b is infinite.
Let f and g two non negative functions, integrable on any interval $[a,c]$ for $a<c<b$.
Assume that for any $t\in[a,b)$, $f(t)\leqslant g(t)$

- If $\displaystyle\int_a^b f(t)\mathrm{d}t$ is divergent, then $\displaystyle\int_a^b g(t)\mathrm{d}t$ is divergent.
- If $\displaystyle\int_a^b g(t)\mathrm{d}t$ is convergent, then $\displaystyle\int_a^b f(t)\mathrm{d}t$ is convergent.

Theorem 23. Let b be a finite number or b is infinite.
Let f and g to positive functions, integrable on any interval $[a,c]$ for $a<c<b$.
If $\displaystyle\lim_{x\to b}\frac{f(x)}{g(x)} = L \neq 0$ (L is a finite number), then $\displaystyle\int_a^b f(t)\mathrm{d}t$ and $\displaystyle\int_a^b g(t)\mathrm{d}t$ are both convergent or both divergent.

3 Absolute convergence

Definition 24. Let b be a finite number or b is infinite.
Let f be a function integrable on any interval $[a,c]$ for $a<c<b$.
The integral $\displaystyle\int_a^b f(t)\mathrm{d}t$ is absolutely convergent if the integral $\displaystyle\int_a^b |f(t)|\,\mathrm{d}t$ is convergent.

Theorem 25. Let b be a finite number or b is infinite.
Let f be a function integrable on any interval $[a, c]$ for $a < c < b$.
If the integral $\displaystyle\int_a^b f(t)\mathrm{d}t$ is absolutely convergent, then $\displaystyle\int_a^b f(t)\mathrm{d}t$ is convergent.

Theorem 26. Let f and Φ be 2 functions such that

- f is continuous on $[a, \infty)$ and $F(x) = \int_a^x f(t)\mathrm{d}t$ is bounded on $[a, \infty)$.
- Φ is differentiable, Φ' is continuous on $[a, \infty)$ and $\lim_{t\to\infty} \Phi(t) = 0$

Then $\displaystyle\int_a^\infty f(t)\Phi(t)\mathrm{d}t$ is convergent.

3.1 Functions defined by improper integrals

We consider functions defined by $\int_a^b f(x,t)\mathrm{d}t$ where b is a finite number or $b = \infty$.

Definition 27.

- The integral $\displaystyle\int_a^b f(x,t)\mathrm{d}t$ is convergent for x in an interval I if $\forall x \in I, \displaystyle\lim_{c\to b} \int_a^c f(x,t)\mathrm{d}t$ exists and is $\displaystyle\int_a^b f(x,t)\mathrm{d}t$ i.e.

$$\forall \mathbf{x} \in \mathbf{I}, \forall \epsilon > 0 \exists c_0, \text{ if } b > c > c_0, \text{ then } \left| \int_a^c f(x,t)\mathrm{d}t - \int_a^b f(x,t)\mathrm{d}t \right| < \epsilon$$

- The integral $\displaystyle\int_a^b f(x,t)\mathrm{d}t$ is uniformly convergent on the interval I if

$$\forall \epsilon > 0 \exists c_0, \text{ if } b > c > c_0, \text{ then } \forall \mathbf{x} \in \mathbf{I} \left| \int_a^c f(x,t)\mathrm{d}t - \int_a^b f(x,t)\mathrm{d}t \right| < \epsilon$$

Theorem 28. If for any $x \in I$, for any $t \in [a, b)$, $|f(t,x)| \leqslant g(t)$ and $\int_a^b g(t)\mathrm{d}t$ is convergent then $\int_a^b f(x,t)\mathrm{d}t$ is uniformly convergent.

Theorem 29. Let $f(x,t)$ be a function continuous of the 2 variables on $I \times [a, b)$. If $\displaystyle\int_a^b f(x,t)\mathrm{d}t$ is uniformly convergent, then $\int_a^b f(x,t)\mathrm{d}t$ is a continuous function of x on the interval $[a, b)$.

Theorem 30. Let $f(x,t)$ be a function continuous of the 2 variables on $I \times [a, b)$. If $\displaystyle\int_a^b f(x,t)\mathrm{d}t$ is uniformly convergent, then for any m and M in I, $\displaystyle\int_m^M \int_a^b f(x,t)\mathrm{d}t = \int_a^b \int_m^M f(x,t)\mathrm{d}t$

Theorem 31. Let $f(x,t)$ be a function continuous of the 2 variables on $I \times [a, b)$.

- If $F(x) = \int_a^b f(t,x)\mathrm{d}t$ is convergent.
- $\dfrac{\partial f}{\partial x}$ exists and is continuous of the 2 variables on $I \times [a, b)$.
- $\displaystyle\int_a^b \frac{\partial f}{\partial x}(x,t)\mathrm{d}t$ is uniformly convergent on I

then $F(x) = \displaystyle\int_a^b f(t,x)\mathrm{d}t$ is differentiable on I and $F'(x) = \displaystyle\int_a^b \frac{\partial f}{\partial x}(x,t)\ dt$.

3.2 Gamma function

Definition 32. The Γ function is defined by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \mathrm{d}t.$$

Theorem 33.

- The function Γ is convergent on $(0, \infty)$.
- For any $x > 0$, $\Gamma(x+1) = x\Gamma(x)$.
- A consequence of the previous property: for any $n \in \mathbb{N}, \Gamma(n) = (n-1)!$.
- Γ is uniformly convergent on any bounded interval $[a, b] \subset (0, \infty)$ but is not uniformly convergent on $(0, \infty)$.
- Γ is continuous, differentiable on $(0, \infty)$.
- Γ is analytic at any $x \in (0, \infty)$. (not proven in class)
- Γ is an analytic extension of the function $(n-1)!$.