University of Washington
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Last Name (PRINT): ______________

First Name (PRINT): ______________

Summer 2015 – Introductory Real Analysis II
Second Examination

Instructions

1. The use of all electronic devices and any additional resources is prohibited. Failure to comply may result in terminating your midterm early
2. Present your solutions in the space provided. Show all your work neatly and concisely. You will be graded not merely on the final answer, but also on the quality and correctness of the work leading up to it.
3. Good luck.

Scholastic dishonesty will not be tolerated.
The work on this test is my own.

Signature:______________

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Grade:

Exercise 1. Let g and f be two functions defined on $\mathbb{R}$.

Assume that for any x, $g(x) > 0$ and that $\lim_{x \to 1} \frac{f(x)}{g(x)} = L \neq 0$.

Prove that $\lim_{x \to 1} f(x) = 0$ iff $\lim_{x \to 1} g(x) = 0$.

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Exercise 2. Prove Theorem 6: Let f be a function defined on a neighborhood of a and f continuous at a.
Let g be a function defined on a neighborhood of $f(a)$ and continuous at $f(a)$.
Prove that $g \circ f$ is continuous at a.
You may use any restult stated prior to Theorem 6.

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Exercise 3. Prove if A is compact and B is closed, then $A \cap B$ is compact.

Exercise 4. Let f be a positive continuous function.
Assume that $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 0$ and $f(0) = 1$,

1. Prove that there exists $\alpha > 0$ such that if $|x| > \alpha$, then $0 < f(x) < \frac{1}{2}$.
2. Prove that f is bounded and f has a maximum.
   $\exists c$ *such that* $f(c) = sup(f)$.

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Exercise 5. Prove that the function $f(x) = \dfrac{1}{x}$ is uniformly continuous on $[1, \infty)$.

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Exercise 6. Let f be an increasing function defined on $[0,1]$. Assume that f is bounded on $[0,1]$.
Prove that f is integrable.

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1 Limits

Definition 1. Let f be a function defined on a neighborhood of a number a except possibly at a,

- $\lim_{x \to a} f(x) = L$ if $\forall \epsilon > 0, \exists \alpha > 0,$ if $|x - a| < \alpha,$ then $|f(x) - f(a)| < \epsilon.$
- $\lim_{x \to a} f(x) = \infty$ if $\forall A > 0, \exists \alpha > 0,$ if $|x - a| < \alpha,$ then $f(x) > A.$

If f is defined on a neighborhood of ∞, (b, ∞)

- $\lim_{x \to \infty} f(x) = L$ if $\forall \epsilon > 0, \exists X > b,$ if $x > X,$ then $|f(x) - f(a)| < \epsilon.$
- $\lim_{x \to \infty} f(x) = \infty$ if $\forall A > 0, \exists X > b,$ if $x > X,$ then $f(x) > A.$

Theorem 2. Let f and g be two functions defined on a neighborhood of a real number a except possibly at a, that have a limit at a.

- $\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x).$
- $\lim_{x \to a} (c. * (x)) = c * \lim_{x \to a} f(x)$ where c is a real number.
- $\lim_{x \to a} (f(x) * g(x)) = \lim_{x \to a} f(x) * \lim_{x \to a} g(x).$
- $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ if $\lim_{x \to a} g(x) \neq 0.$

A similar theorem holds for limits at infinity.

Theorem 3. (Squeeze theorem)
Let f, g, h be 3 functions defined on a neighborhood of a. If $f \leqslant g \leqslant h$ on a neighborhood of a and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = l$, then $\lim_{x \to a} g(x)$ exists and $\lim_{x \to a} g(x) = l$

2 Continuity

Definition 4. Let f be a function defined on a neighborhood of a number a, the function f is continuous at a if

$$\lim_{x \to a} f(x) = f(a)$$

Theorem 5. Let f and g be 2 functions defined on a neighborhood of a and continuous at a. then $f + g$, $f - g$, $f * g$, $\dfrac{f}{g}$ is continuous at a provided that $g(a) \neq 0$ for the quotient.

Theorem 6. If f be a function defined on a neighborhood of a and continuous at a, and g is a function defined in a neighborhood of $f(a)$ and continuous at $f(a)$ then Then $g \circ f$ is continuous at a.

3 Differentiability

Definition 7. Let f be a function defined on a neighborhood of a.

If the limit $\lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}$ exists and is finite, then the function f is differentiable at a and the derivative at a of f, written $f'(a)$ or $\dfrac{\mathrm{d}f}{dx}(a)$ is $\lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}$.

Theorem 8. If f is differentiable at a, then f is continuous at a.

Theorem 9. (Chain rule) If f is a function differentiable at a and g is a function differentiable at $f(a)$, then $g \circ f$ is differentiable at a and $(g \circ f)'(a) = f'(a) * g'(f(a))$.

4 Heine Borel Theorem

Definition 10. A collection of open sets $\mathcal{A} = \{\mathcal{U}_\alpha, \alpha \in I\}$ is a cover by open sets of a set S if $S \subset \cup_{\alpha \in I} \mathcal{U}_\alpha$.
A subcover is a subset of $\mathcal{A}$ that covers S.

Definition 11. A set S is compact if whenever S is covered by open sets $\{\mathcal{U}_\alpha, \alpha \in I\}$, there exists a finite subcover of S.

Theorem 12. (Heine-Borel Theorem) A set S is compact iff S is closed and bounded.

5 Theorems on continuous functions

Theorem 13. Let f be a function defined on a neighborhood of p.
f is continuous at p iff for any sequence x_n convergent to p, the sequence $f(x_n)$ is convergent to $f(p)$.

Theorem 14. Let S be a compact set.
If f is continuous on S, then $f(S)$ is compact.

Theorem 15. (Extreme Value Theorem) Let S be a non empty closed and bounded set (compact). and a function f continuous on S.
If M, m are the least upper bound and greatest lower bound respectively, there exists x_M and x_m in S such that $f(x_M) = M$ and $f(x_m) = m$.

Theorem 16. (Intermediate Value Theorem) Let f be continuous on an interval $[a, b]$. Assume that $f(a) \neq f(b)$,
For any y between $f(a)$ and $f(b)$, there exists a real c in (a, b) such that $f(c) = y$.

6 Uniform continuity

Definition 17. Let f be a function defined on an interval I.
f is continuous on I iff

$$\forall x \in I, \forall \epsilon > 0, \exists \alpha > 0, \text{ if } |x - t| < \alpha \implies |f(x) - f(t)| < \epsilon$$

f is uniformly continuous on I iff

$$\forall \epsilon > 0, \exists \alpha > 0, \forall x \in I \text{ if } |x - t| < \alpha \implies |f(x) - f(t)| < \epsilon$$

Example: The function $f(x) = x^2$ is uniformly continuous on $[0, 5)$ but is not uniformly continuous on $\mathbb{R}$.

Theorem 18. If S is compact (closed and bounded) and f is a continuous function on S
then f is uniformly continuous on S.

7 Theory of integration

Definition 19. Given an interval $[a, b]$, a partition of $[a, b]$ is a finite collection of real numbers $\{x_i\}$ such that $a = x_0 < x_1 < x_2 < \cdots < x_n = b$

Definition 20. Let f be a bounded function on $[a, b]$, $\{x_i\}$ be a partition of $[a, b]$.
Let M_i and m_i be the least upper bound and greatest lower bound of f on the interval $[x_i, x_{i+1}]$.
We define the Riemann sums by

$$S = \sum_{i=0}^{n-1} (x_{i+1} - x_i) M_i, \qquad \text{and} \qquad s = \sum_{i=0}^{n-1} (x_{i+1} - x_i) m_i$$

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Proposition 21. When adding points to the partition, the Riemann sums S are decreasing and s are increasing.

Proposition 22. Let C_1 and C_2 be two partions of $[a, b]$. If S_1 and s_1 are the Rieman sums corresponding to the partition C_1 ,and S_2 and s_2 are the Riemann sums corresponding to C_2, then $S_1 \geqslant s_2$

Definition 23. Let I be the greatest lower bound of S for all the possible partitions of $[a, b]$ and J be the least upper bound of s for all the possible partitions of $[a, b]$.

Proposition 24. $I \geqslant J$

Definition 25. Given a bounded function f on an interval $[a, b]$ If $I = J$, then the function f is integrable on $[a, b]$ and the definite integral $\displaystyle\int_a^b f(t)\mathrm{d}t = I = J$

Theorem 26. A function f is integrable iff for any $\epsilon > 0$ there exists a partition such that $S - s < \epsilon$.

Theorem 27. If f is continuous on $[a, b]$, then f is integrable on $[a, b]$

Theorem 28. (Theorem fundamental of calculus)
If f is continuous on $[a, b]$ then the function $F(x) = \displaystyle\int_a^x f(t)\mathrm{d}t$ is differentiable on $[a, b]$ and $F'(x) = f(x)$.

Theorem 29. Let f be a contiuous non negative function on an interval $[a, b]$.
If $\int_a^b f(t)]dt = 0$, then the function f is 0 on the entire interval $[a, b]$.