University of Washington

Last Name (PRINT): ____________________
First Name (PRINT): ____________________

Summer 2015 – Introductory Real Analysis II
Final Examination

Instructions

1. The use of all electronic devices and any additional resources is prohibited. Failure to comply may result in terminating your midterm early
2. Present your solutions in the space provided. Show all your work neatly and concisely. You will be graded not merely on the final answer, but also on the quality and correctness of the work leading up to it.
3. Good luck.

Scholastic dishonesty will not be tolerated.
The work on this test is my own.

Signature:____________________

Grade:

Exercise 1. Find the radius R of convergence of the power series

$$S(x) = \sum_{n=1}^{\infty} \frac{x^{4n-1}}{4n}$$

and evaluate the sum $S(x)$ at any x in the open disc of convergence $(-R, R)$.

Exercise 2. Determine whether the integral

$$\int_0^\infty \frac{t^5}{(t^4+1)\sqrt{t}}\mathrm{d}t$$

is convergent.

Exercise 3. Given the function

$$F(x) = \int_0^\infty \frac{e^{-xt}}{1+t^2}\mathrm{d}t$$

1. For which values of x, F is convergent?
2. Prove that F is continuous on $[0, \infty)$.

Exercise 4. Assume that f is a continuous function on $\mathbb{R}$ and $\lim_{x\to-\infty} f(x) = -1$ and $\lim_{x\to\infty} f(x) = 1$, prove that there exists a real c such that $f(c) = 0$.

Exercise 5. Prove that if f is uniformly continuous on $[a, b]$ then f is integrable on $[a, b]$.

This is theorem 60, you may use any theorem metioned above theorem 60

1 Power Series

1.1 Radius of convergence

Definition 1. A power series is a series of functions in the form $f(x) = \sum_{k=0}^{\infty} a_n x^n$.

Theorem 2. Given a power series $f(x) = \sum_{n=-}^{\infty} a_n x^n$, if there exists x_0 such that the series $f(x_0)$ is convergent, then for any x such that $|x| < |x_o|$, the series $f(x)$ is absolutely convergent.

Theorem 3. Given a power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$, exactly one of the 3 statements is true

- f is convergent only for $x = 0$. We say that the radius of convergence is 0.
- f is convergent for any x. We say that the radius of convergence is infinite.
- there exists a positive real R, called radius of convergence such that
  - if $|x| < R$, then the series is absolutely convergent.
  - if $|x| > R$, then the series is divergent.

Theorem 4. Let f be a power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$.

If the limit $\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$ exists or is infinite, then the radius of convergence is $R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$.

1.2 Uniform convergence of power series

Theorem 5. Let f be a power series with a positive or infinite radius of convergence R.
For any $0 < r < R$, the power series is uniformly convergent on $[-r, r]$.

Theorem 6. Let $f = \sum_{n=0}^{\infty} a_n x^n$ be a power series with a positive or infinite radius of convergence R.
The series f is continuous on the open interval $(-R, R)$.
Moreover, for any a and b in the open interval $(-R, R)$,

$$\int_a^b f(x)\mathrm{d}x = \sum_{n=0}^{\infty} a_n \left(\frac{b^{n+1} - a^{n+1}}{n+1}\right)$$

1.3 Differentiability of power series

Theorem 7. The power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ have the same radius of convergence.

Theorem 8. Let $f = \sum_{n=0}^{\infty} a_n x^n$ be a power series with positive or infinite radius of convergence R.

Then f is differentiable at any point in open interval $(-R, R)$ and its derivative is $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$.
By induction, f is differentiable infinitely many times on the open disk of convergence.
And, $a_n = \frac{f^{(n)}(0)}{n!}$.

Theorem 9. If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series with a positive radius of convergence, then $a_n = \dfrac{f^{(n)}(0)}{n!}$

Theorem 10. If $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^2$ on a neighborhood of 0, then for any n, $a_n = b_n$.

1.4 Product and quotient of power series

Definition 11. If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ are 2 powers series with 2 positive radius of convergence R_a and R_b, Then the product of the power series is a power series $h(x0 = \sum_{n=0}^{\infty} c_n x^n$ with $c_n = \sum_{k=0}^{n} a_k b_{n-k}$

Its radius of convergence R is greater of equal to $\min(R_a, R_b)$.

Theorem 12. Given 2 power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ with $b_0 \neq 0$, then $\dfrac{\sum a_n x^n}{\sum b_n x_n}$ can be represented by a power series $\sum c_n x^n$.

To find the power series $\sum c_n x^n$, you may use a long division or notice that $\sum a_n x^n = \left(\sum b_n x^n\right)\left(\sum c_n x^n\right)$. The latest method leads to a triangular system of equations

$$\begin{aligned}
a_0 &= b_0 c_0 & &\implies c_0 = \cdots \\
a_1 &= b_0 c_1 + b_1 c_0 & &\implies c_1 = \cdots \\
&\vdots & &\vdots \\
a_n &= b_0 c_n + b_1 c_{n-1} + \cdots + b_{n-1} c_1 + b_n c_0 & &\implies c_n = \cdots
\end{aligned}$$

A third method involves the geometric series formula $\sum_{n=0}^{\infty} x^n = \dfrac{1}{1-x}$.

For example, the function $\dfrac{1}{1+x^2}$ can be represented by the power series $\dfrac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$.

1.5 Inferior limit, superior limit

Definition 13. Given a sequence u_n,

$$\overline{\lim_{n\to\infty}} x_n = \lim_{n\to\infty} \left(\sup\{x_k; k \geqslant n\}\right)$$

$$\underline{\lim} x_n = \lim_{n\to\infty} \left(\inf\{x_k; k \geqslant n\}\right)$$

Example: If $x_n = (-1)^n \left(1 + \dfrac{1}{n}\right)$, then $\overline{\lim} x_n = 1$ and $\underline{\lim} x_n = -1$

Theorem 14. Given a power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$, then its radius of convergence R satisfy

$$\frac{1}{R} = \overline{\lim_{n\to\infty}} |a_n|^{1/n}$$

Example: The radius of convergence of the series $\sum_{n=0}^{\infty} \dfrac{x^{n^2}}{2^n}$ is 1.

1.6 Analytic function

Definition 15. Let f be a real valued function defined on (a,b) such that f has derivative of all order at any point x in (a,b).

The function f is analytic on (a,b) is for any x_0 in (a,b), there exists a neighborhood of x_0 such that

$$f(x)=\sum_{k=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

on the neighborhood of x_0.

Example

- $e^x=\sum_{k=0}^{\infty}$ is analytic on $\mathbb{R}$. Using Euler formula, $\cos x$. $\sin x$, $\tan x$ are analytic on $\mathbb{R}$.
- $\dfrac{1}{1-x}$ is analytic on $(-\infty,1)$, and on $(1,\infty)$.
  Therefore $\operatorname{Arctan} x$ is analytic on $\mathbb{R}$, $\ln(1+x)$ is analytic on $(-1,\infty)$...
- The function $x^{5/2}$ is not analytic at 0 since $x^{5/2}$ does not have derivatives of all orders.
- The function $f(x)=\begin{cases}0 & x=0\\ e^{-1/x^2} & x\neq 0\end{cases}$ has derivatives of all order at $x=0$.

  All the successive derivative at $x=0$ are 0.

  The Taylor series of f, T_f, is 0. On any neighborhood of f, the Taylor series T_f and f are different functions. Therefore f is not analytic at 0.

Theorem 16. Let f be a function defined on (a,b) such that f has derivative of all order on (a,b) and $f^{(n)}(x)\geqslant 0$ at any $x\in(a,b)$.

Then f is analytic on (a,b).

Theorem 17.

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!} \qquad \frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$$

$$\cos x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{2n!} \qquad \ln(1+x)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$$

$$\sin x\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} \qquad \operatorname{Arctan} x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$$

2 Improper integrals

2.1 Improper integral of the first kind and of the second kind

Definition 18. Improper of the first kind

Let f be an integrable function over every finite intervals $[a,c]$ for $a<c$.

- The improper integral of the first kind $\displaystyle\int_a^{\infty} f(x)\mathrm{d}x$ is defined by

$$\int_a^{\infty} f((x)\mathrm{d}x=\lim_{c\to\infty}\int_a^c f(x)\mathrm{d}x$$

  if the limit exists.

- If the limit exists, the improper integral is convergent.
- If the limit does not exists, the improper integral is divergent.

Definition 19. Improper of the second kind
Let f be a function on $[a,b)$, integrable over every finite intervals $[a,c]$ for any c, $a<c<b$.

- The improper integral of the second kind $\displaystyle\int_a^b f(x)\mathrm{d}x$ is defined by

$$\int_a^b f((x)\mathrm{d}x = \lim_{c\to\infty}\int_a^c f(x)\mathrm{d}x$$

  if the limit exists.

- If the limit exists, the improper integral is convergent.
- If the limit does not exists, the improper integral is divergent.

Theorem 20. $\displaystyle\int_0^1 t^\alpha \mathrm{d}t$ is convergent for $\alpha > -1$.
$\displaystyle\int_1^\infty t^\alpha \mathrm{d}t$ is convergent for $\alpha < -1$.

2.2 Improper integral with non negative integrand

Theorem 21. Let b be a finite real number or b is infinite.
Let f be a **non negative** function integrable on any interval $[a,c]$ for $a<c<b$.
The improper integral $\int_a^b f(t)\mathrm{d}t$ is convergent iff $F(x)=\int_a^x f(t)\mathrm{d}t$ is bounded for $x\in[a,b)$.

Theorem 22. Let b be a finite number or b is infinite.
Let f and g two non negative functions, integrable on any interval $[a,c]$ for $a<c<b$.
Assume that for any $t\in[a,b)$, $f(t)\leqslant g(t)$

- If $\displaystyle\int_a^b f(t)\mathrm{d}t$ is divergent, then $\displaystyle\int_a^b g(t)\mathrm{d}t$ is divergent.
- If $\displaystyle\int_a^b g(t)\mathrm{d}t$ is convergent, then $\displaystyle\int_a^b f(t)\mathrm{d}t$ is convergent.

Theorem 23. Let b be a finite number or b is infinite.
Let f and g to positive functions, integrable on any interval $[a,c]$ for $a<c<b$.
If $\displaystyle\lim_{x\to b}\frac{f(x)}{g(x)} = L \neq 0$ (L is a finite number), then $\displaystyle\int_a^b f(t)\mathrm{d}t$ and $\displaystyle\int_a^b g(t)\mathrm{d}t$ are both convergent or both divergent.

3 Absolute convergence

Definition 24. Let b be a finite number or b is infinite.
Let f be a function integrable on any interval $[a,c]$ for $a<c<b$.
The integral $\displaystyle\int_a^b f(t)\mathrm{d}t$ is absolutely convergent if the integral $\displaystyle\int_a^b |f(t)|\,\mathrm{d}t$ is convergent.

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Theorem 25. Let b be a finite number or b is infinite.
Let f be a function integrable on any interval $[a, c]$ for $a < c < b$.
If the integral $\displaystyle\int_a^b f(t)\mathrm{d}t$ is absolutely convergent, then $\displaystyle\int_a^b f(t)\mathrm{d}t$ is convergent.

Theorem 26. Let f and Φ be 2 functions such that

- f is continuous on $[a, \infty)$ and $F(x) = \int_a^x f(t)\mathrm{d}t$ is bounded on $[a, \infty)$.
- Φ is differentiable, Φ' is continuous on $[a, \infty)$ and $\lim_{t\to\infty} \Phi(t) = 0$

Then $\displaystyle\int_a^\infty f(t)\Phi(t)\mathrm{d}t$ is convergent.

3.1 Functions defined by improper integrals

We consider functions defined by $\int_a^b f(x, t)\mathrm{d}t$ where b is a finite number or $b = \infty$.

Definition 27.

- The integral $\displaystyle\int_a^b f(x, t)\mathrm{d}t$ is convergent for x in an interval I if $\forall x \in I, \displaystyle\lim_{c\to b} \int_a^c f(x, t)\mathrm{d}t$ exists and is $\displaystyle\int_a^b f(x, t)\mathrm{d}t$ i.e.

$$\forall \mathbf{x} \in \mathbf{I}, \forall \epsilon > 0 \exists c_0, \text{ if } b > c > c_0, \text{ then } \left| \int_a^c f(x, t)\mathrm{d}t - \int_a^b f(x, t)\mathrm{d}t \right| < \epsilon$$

- The integral $\displaystyle\int_a^b f(x, t)\mathrm{d}t$ is uniformly convergent on the interval I if

$$\forall \epsilon > 0 \exists c_0, \text{ if } b > c > c_0, \text{ then } \forall \mathbf{x} \in \mathbf{I} \left| \int_a^c f(x, t)\mathrm{d}t - \int_a^b f(x, t)\mathrm{d}t \right| < \epsilon$$

Theorem 28. If for any $x \in I$, for any $t \in [a, b)$, $|f(t, x)| \leqslant g(t)$ and $\int_a^b g(t)\mathrm{d}t$ is convergent then $\int_a^b f(x, t)\mathrm{d}t$ is uniformly convergent.

Theorem 29. Let $f(x, t)$ be a function continuous of the 2 variables on $I \times [a, b)$. If $\displaystyle\int_a^b f(x, t)\mathrm{d}t$ is uniformly convergent, then $\int_a^b f(x, t)\mathrm{d}t$ is a continuous function of x on the interval $[a, b)$.

Theorem 30. Let $f(x, t)$ be a function continuous of the 2 variables on $I \times [a, b)$. If $\displaystyle\int_a^b f(x, t)\mathrm{d}t$ is uniformly convergent, then for any m and M in I, $\displaystyle\int_m^M \int_a^b f(x, t)\mathrm{d}t = \int_a^b \int_m^M f(x, t)\mathrm{d}t$

Theorem 31. Let $f(x, t)$ be a function continuous of the 2 variables on $I \times [a, b)$.

- If $F(x) = \int_a^b f(t, x)\mathrm{d}t$ is convergent.
- $\dfrac{\partial f}{\partial x}$ exists and is continuous of the 2 variables on $I \times [a, b)$.
- $\displaystyle\int_a^b \frac{\partial f}{\partial x}(x, t)\mathrm{d}t$ is uniformly convergent on I

then $F(x) = \displaystyle\int_a^b f(t, x)\mathrm{d}t$ is differentiable on I and $F'(x) = \displaystyle\int_a^b \frac{\partial f}{\partial x}(x, t)\ dt$.

3.2 Gamma function

Definition 32. The Γ function is defined by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\mathrm{d}t.$$

Theorem 33.
- The function Γ is convergent on $(0, \infty)$.
- For any $x > 0$, $\Gamma(x+1) = x\Gamma(x)$.
- A consequence of the previous property: for any $n \in \mathbb{N}, \Gamma(n) = (n-1)!$.
- Γ is uniformly convergent on any bounded interval $[a, b] \subset (0, \infty)$ but is not uniformly convergent on $(0, \infty)$.
- Γ is continuous, differentiable on $(0, \infty)$.
- Γ is analytic at any $x \in (0, \infty)$. (not proven in class)
- Γ is an analytic extension of the function $(n-1)!$.

4 Continuity of functions

4.1 Limits

Definition 34. Let f be a function defined on a neighborhood of a number a except possibly at a,

- $\lim\limits_{x\to a} f(x) = L$ if $\forall\epsilon > 0, \exists\alpha > 0,$ if $|x-a| < \alpha,$ then $|f(x) - L| < \epsilon$.
- $\lim\limits_{x\to a} f(x) = \infty$ if $\forall A > 0, \exists\alpha > 0,$ if $|x-a| < \alpha,$ then $f(x) > A$.

If f is defined on a neighborhood of ∞, (b, ∞)

- $\lim\limits_{x\to\infty} f(x) = L$ if $\forall\epsilon > 0, \exists X > b,$ if $x > X,$ then $|f(x) - L| < \epsilon$.
- $\lim\limits_{x\to\infty} f(x) = \infty$ if $\forall A > 0, \exists X > b,$ if $x > X,$ then $f(x) > A$.

Theorem 35. Let f and g be two functions defined on a neighborhood of a real number a except possibly at a, that have a limit at a.

- $\lim\limits_{x\to a}(f(x) + g(x)) = \lim\limits_{x\to a} f(x) + \lim\limits_{x\to a} g(x)$.
- $\lim\limits_{x\to a}(c. * (x)) = c * \lim\limits_{x\to a} f(x)$ where c is a real number.
- $\lim\limits_{x\to a}(f(x) * g(x)) = \lim\limits_{x\to a} f(x) * \lim\limits_{x\to a} g(x)$.
- $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$ if $\lim\limits_{x\to a} g(x) \neq 0$.

A similar theorem holds for limits at infinity.

Theorem 36. (Squeeze theorem)
Let f, g, h be 3 functions defined on a neighborhood of a. If $f \leqslant g \leqslant h$ on a neighborhood of a and $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} h(x) = l$, then $\lim\limits_{x\to a} g(x)$ exists and $\lim\limits_{x\to a} g(x) = l$

4.2 Continuity

Definition 37. Let f be a function defined on a neighborhood of a number a, the function f is continuous at a if

$$\lim_{x\to a} f(x) = f(a)$$

Theorem 38. Let f and g be 2 functions defined on a neighborhood of a and continuous at a. then $f+g$, $f-g$, $f*g$, $\dfrac{f}{g}$ is continuous at a provided that $g(a)\neq 0$ for the quotient.

Theorem 39. If f be a function defined on a neighborhood of a and continuous at a, and g is a function defined in a neighborhood of $f(a)$ and continuous at $f(a)$ then Then $g\circ f$ is continuous at a.

4.3 Differentiability

Definition 40. Let f be a function defined on a neighborhood of a.

If the limit $\lim_{x\to a}\dfrac{f(x)-f(a)}{x-a}$ exists and is finite, then the function f is differentiable at a and the derivative at a of f, written $f'(a)$ or $\dfrac{\mathrm{d}f}{dx}(a)$ is $\lim_{x\to a}\dfrac{f(x)-f(a)}{x-a}$.

Theorem 41. If f is differentiable at a, then f is continuous at a.

Theorem 42. (Chain rule) If f is a function differentiable at a and g is a function differentiable at $f(a)$, then $g\circ f$ is differentiable at a and $(g\circ f)'(a) = f'(a)*g'(f(a))$.

4.4 Heine Borel Theorem

Definition 43. A collection of open sets $\mathcal{A} = \{\mathcal{U}_\alpha, \alpha\in I\}$ is a cover by open sets of a set S if $S\subset \cup_{\alpha\in I}\mathcal{U}_\alpha$.

A subcover is a subset of $\mathcal{A}$ that covers S.

Definition 44. A set S is compact if whenever S is covered by open sets $\{\mathcal{U}_\alpha, \alpha\in I\}$, there exists a finite subcover of S.

Theorem 45. (Heine-Borel Theorem) A set S is compact iff S is closed and bounded.

4.5 Theorems on continuous functions

Theorem 46. Let f be a function defined on a neighborhood of p.

f is continuous at p iff for any sequence x_n convergent to p, the sequence $f(x_n)$ is convergent to $f(p)$.

Theorem 47. Let S be a compact set.

If f is continuous on S, then $f(S)$ is compact.

Theorem 48. (Extreme Value Theorem) Let S be a non empty closed and bounded set (compact). and a function f continuous on S.

If M, m are the least upper bound and greatest lower bound respectively, there exists x_M and x_m in S such that $f(x_M) = M$ and $f(x_m) = m$.

Theorem 49. (Intermediate Value Theorem) Let f be continuous on an interval $[a,b]$. Assume that $f(a)\neq f(b)$,

For any y between $f(a)$ and $f(b)$, there exists a real c in (a,b) such that $f(c)=y$.

4.6 Uniform continuity

Definition 50. Let f be a function defined on an interval I.
f is continuous on I iff

$$\forall x \in I, \forall \epsilon > 0, \exists \alpha > 0, \text{ if } |x - t| < \alpha \implies |f(x) - f(t)| < \epsilon$$

f is uniformly continuous on I iff

$$\forall \epsilon > 0, \exists \alpha > 0, \forall x \in I \text{ if } |x - t| < \alpha \implies |f(x) - f(t)| < \epsilon$$

Example: The function $f(x) = x^2$ is uniformly continuous on $[0, 5)$ but is not uniformly continuous on $\mathbb{R}$.

Theorem 51. If S is compact (closed and bounded) and f is a continuous function on S then f is uniformly continuous on S.

5 Theory of integration

Definition 52. Given an interval $[a, b]$, a partition of $[a, b]$ is a finite collection of real numbers $\{x_i\}$ such that $a = x_0 < x_1 < x_2 < \cdots < x_n = b$

Definition 53. Let f be a bounded function on $[a, b]$, $\{x_i\}$ be a partition of $[a, b]$.
Let M_i and m_i be the least upper bound and greatest lower bound of f on the interval $[x_i, x_{i+1}]$.
We define the Riemann sums by

$$S = \sum_{i=0}^{n-1} (x_{i+1} - x_i) M_i, \qquad \text{and} \qquad s = \sum_{i=0}^{n-1} (x_{i+1} - x_i) m_i$$

Proposition 54. When adding points to the partition, the Riemann sums S are decreasing and s are increasing.

Proposition 55. Let C_1 and C_2 be two partions of $[a, b]$. If S_1 and s_1 are the Rieman sums corresponding to the partition C_1 ,and S_2 and s_2 are the Riemann sums corresponding to C_2, then $S_1 \geqslant s_2$

Definition 56. Let I be the greatest lower bound of S for all the possible partitions of $[a, b]$ and J be the least upper bound of s for all the possible partitions of $[a, b]$.

Proposition 57. $I \geqslant J$

Definition 58. Given a bounded function f on an interval $[a, b]$ If $I = J$, then the function f is integrable on $[a, b]$ and the definite integral $\displaystyle\int_a^b f(t)\mathrm{d}t = I = J$

Theorem 59. A function f is integrable iff for any $\epsilon > 0$ there exists a partition such that $S - s < \epsilon$.

Theorem 60. If f is continuous on $[a, b]$, then f is integrable on $[a, b]$

Theorem 61. (Theorem fundamental of calculus)
If f is continuous on $[a, b]$ then the function $F(x) = \displaystyle\int_a^x f(t)\mathrm{d}t$ is differentiable on $[a, b]$ and $F'(x) = f(x)$.

Theorem 62. Let f be a contiuous non negative function on an interval $[a, b]$.
If $\int_a^b f(t)]dt = 0$, then the function f is 0 on the entire interval $[a, b]$.