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1 Axioms of $\mathbb{R}$

Axiom 1. $\mathbb{R}$ is a commutative field.

Definition 1. $F + *$ is a commutative field if

1. For any a, b, c in F, $a + (b + c) = (a + b) + c$, and $a * (b * c) = (a * b) * c$ (associativity).
2. For any a and b in F, $a + b = b + a$ and $a * b = b * a$ (commutativity).
3. For any a, b, c in F, $a * (b + c) = a * b + a * c$ (distributivity).
4. There exists an element written 0 such that for any a in F, $a + 0 = a$ (additive identity)
5. There exists an element written 1 such that for any a in F, $a * 1 = a$ (multiplicative identity)
6. For any a in F, there exists an element written $-a$ such that $a+(-a) = 0$ (additive inverse)
7. For any a in F except 0, there exists an element written a^{-1} such that $a * (a^{-1}) = 1$ (multiplicative inverse).

Proposition 2.

1. The additive inverse is unique.
2. For any real number a, $0 * a = 0$.
3. $(-1)(-1) = 1$.
4. The additive inverse of $a + b$ is $(-a) + (-b)$.
5. for any real number a, $-a = (-1) * a$.

Axiom 2. $\mathbb{R}$ is totally ordered:
There is a relation $>$ that satisfies

1. If $a > 0$ and $b > 0$, then $a + b > 0$
2. If $a > 0$ and $b > 0$, then $a * b > 0$
3. For each a only one of the following is true
   (a) $a > 0$,
   (b) $a = 0$,
   (c) $-a > 0$

Definition 3. We say that $a < b$ if $b - a > 0$

Proposition 4. Given three real numbers a, b, c,

1. $0 < a^2$ if $a \neq 0$
2. If $a < b$ and $b < c$ then $a < c$
3. If $a < b$ and $0 < c$, then $ac < bc$
4. If $a < b$, for any c, $a + c < b + c$

Proposition 5.
1. If a is a positive real number, then its multiplicative inverse a^{-1} is positive.
2. For any real numbers such that $0 < a < b$, then $b^{-1} < a^{-1}$.

Definition 6. The absolute value of a, written $|a|$ is

- $|a| = a$ if $a > 0$
- $|a| = -a$ if $a < 0$
- $|a| = 0$ if $a = 0$

Proposition 7. For any a, b in $\mathbb{R}$,

- $|a * b| = |a| * |b|$
- $|a + b| \leqslant |a| + |b|$
- $||a| - |b|| \leqslant |a - b|$

Proposition 8. For any real numbers such that $a < b < c$, $|b| \leqslant max\left(|a|, |c|\right)$.

Axiom 3. (Axiom of continuity) Suppose that all real numbers are separated into two collections which we denote by L and R, in such a way that

1. Every number is either in L or in R.
2. Each collection contains at least 1 element.
3. If a is in L and b is in R, then $a < b$.

then there is a number c in $\mathbb{R}$ such that all numbers less than c are in L and all numbers greater that c are in R.

Proposition 9. The cut number c is unique.

Theorem 10. (Archimedian law of real numbers) Let a and b be 2 positive real numbers. There exists a positive integer n such that $b < na$.

Proposition 11. If a real number y satisfies that $0 \leqslant y \leqslant \frac{1}{n}$ for any natural number n, then $y = 0$.

Proposition 12. For any positive rel number y, there exists a real number c such that $c^2 = y$.

Theorem 13. Any system that satisfies Axiom 1, 2, and 3 is isomorphic to $\mathbb{R}$

2 Axioms of $\mathbb{N}$ (2.3)

Axiom 4. The natural numbers are the smallest class of real numbers that satisfies

1. 1 is an element of $\mathbb{N}$
2. If n is an element of $\mathbb{N}$, $n + 1$ is an element of $\mathbb{N}$

Proposition 14. Any natural number either equals to 1 or is greater than 1.

Proposition 15. For any 2 natural numbers $p < n$, there exist a natural number m such that $n = p + m$.

Proposition 16. Let n be a natural number. there is no natural number between n and $n + 1$.

Proposition 17. If S is a class of positive integers containing at least 1 element, it contains a smallest element.

3 Rational & irrational numbers (2.5)

Definition 18. A rational number is a number that can be written in the form $\pm\frac{p}{q}$, where p and q are 2 natural numbers.
An irrational number is a number that is not rational

Theorem 19. The number $\sqrt{2}$ is irrational.

Theorem 20. Given to numbers $a < b$, there is a rational number between a and b.
There is an irrational number between a and b.

4 Least upper bounds, greatest lower bounds (2.6)

Theorem 21. If S is a set of real numbers which is not empty and which has an upper bound, then it has a least upper bound.

Theorem 22. The greatest lower bound is unique whenever it exists.

Theorem 23. If S is a set of real numbers which is not empty and which has a lower bound, then it has a greatest lower bound.

Theorem 24. Given two non empty subsets of $\mathbb{R}$, A and B which have two lower bounds. If $A + B$ is defined by

$$A + B = \{a + b, a \in A, b \in B\}$$

then,

$$inf(A + B) = inf(A) + inf(B)$$

Similar results holds for least upper bounds and can be used without proof.

Proposition 25. If A and B are 2 non empty subsets of $\mathbb{R}$,

1. If $-A$ is the set of all additive inverses of elements of A, then $\sup(-A) = -\inf(A)$.
2. If A is a subset of B, then $\inf(B) \leqslant \inf(A) \leqslant \sup(A) \leqslant \sup(B)$.

5 Limits of sequences

Definition 26. A sequence of real number is a function from $\mathbb{N}$ to $\mathbb{R}$ whose domain is $\mathbb{N}$.

Definition 27. Given a sequence of real number s_n, we say that $\lim_{n\to\infty} s_n = l$ if for each positive number ϵ, there is some integer N (depending on ϵ) such that if $n > N$, then $|s_n - l| < \epsilon$.
In this case we say that s_n is convergent.
We say that $\lim_{n\to\infty} s_n = \infty$ is for any number A, there is some integer N (depending on A) such that if $n > N$, then $s_n > A$.

Proposition 28.
$$\lim_{n\to\infty} \left(\frac{1}{n}\right) = 0$$

$$\lim_{n\to\infty} a^n = 0 \qquad \text{if} \quad 0 < a < 1$$
$$\lim_{n\to\infty} a^n = \infty \qquad \text{if} \quad 1 < a$$

Theorem 29. Given two convergent sequences s_n and t_n, then

- For any real numbers a and b, then
$$\lim_{n\to\infty} (as_n + bt_n) = a\left(\lim_{s\to\infty} s_n\right) + b\left(\lim_{s\to\infty} t_n\right).$$
- $\lim_{n\to\infty} (s_n t_n) = \left(\lim_{n\to\infty} s_n\right)\left(\lim_{n\to\infty} t_n\right)$.
- If $\left(\lim_{n\to\infty} t_n\right) \neq 0$, then $\lim_{n\to\infty} \left(\frac{s_n}{t_n}\right) = \frac{\lim_{n\to\infty} s_n}{\lim_{n\to\infty} t_n}$.

Theorem 30. Given a sequence of real number s_n,

- If, for any $n > N$ for some natural number N, $s_{n+1} \leqslant s_n$ and if there exists a real M such that $s_n \geqslant M$ for any n then s_n is convergent and $\lim_{n\to\infty} s_n \geqslant M$.
- If, for any $n > N$ for some natural number N, $s_{n+1} \geqslant s_n$ and if there exists a real M such that $s_n \leqslant M$ for any n then s_n is convergent and $\lim_{n\to\infty} s_n \leqslant M$.

Theorem 31. Squeeze theorem Given 3 sequences s_n, t_n and v_n, if there exists a natural number N

- for any $n > N$, $s_n \leqslant t_n \leqslant v_n$
- $\lim_{n\to\infty} s_n = \lim_{n\to\infty} v_n = l$

Then t_n is convergent and $\lim_{n\to\infty} t_n = l$.

Theorem 32. Given 2 sequences s_n and t_n, assume $s_n \leqslant t_n$ for any $n > N$ for some N

- If $\lim_{n\to\infty} s_n = \infty$ then $\lim_{n\to\infty} t_n = \infty$.
- If $\lim_{n\to\infty} t_n = -\infty$, then $\lim_{n\to\infty} s_n = -\infty$.

Theorem 33. Nested Intervals (2.8) Given a sequence of nested closed intervals $I_n = [a_n, b_n]$, $(a_n \leqslant a_{n+1}$, and $b_n \geqslant b_n + 1)$, their intersection is either a singleton or a closed interval.

6 Chapter 16

6.1 Open sets, Closed sets

Definition 34. A neighborhood of x_0 is a set of points x such that $|x - x_0| < h$ or equivalently $(x_0 - h, x_0 + h)$ for some positive real h.

Definition 35. A set S is open if any element s in S has a neighborhood $(s - h, s + h)$ entirely included in S.
A set S is closed if its complementary is open.

Remark: the sets

- $\emptyset$ and $(-\infty, \infty)$ are both open and closed.
- (a, b) is open with a being possibly $-\infty$ and b being ∞.
- $\mathbb{R} \setminus \{a\}$ is open for any real a.
- $[a, b]$, $(-\infty, b]$, $[a, \infty)$, $\{a\}$ are closed.
- The set $\mathbb{R} \setminus \left\{1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots\right\}$ is neither closed nor open.

Theorem 36. If A and B are two open sets, then

- $A \cup B$ is open
- $A \cap B$ is open.

If A and B are closed sets, then

- $A \cup B$ is closed,
- $A \cap B$ is closed.

6.2 Accumulation points

Definition 37. A element y is an accumulation point for a set S if any neighborhood of y contains at least one element of S that is not y. i. e for any h the intersection $S \cap (y - h, y + h) \setminus \{y\}$ is not empty.

Theorem 38. A set S is closed if and only if any accumulation point of S is in S.

Theorem 39. (Bolzano Weierstrass) Any infinite bounded set has at least one accumulation point.

Theorem 40. An element y is an accumulation of a set S iff there exists a sequence of distinct elements of S that is convergent to y.

Theorem 41. Given a convergent sequence s_n of real numbers that takes infinitely many distinct values, the set $S = \{s_n, n \in \mathbb{N}\}$ has exactly one accumulation point, the limit of s_n.

Theorem 42. Let s_n be a bounded sequence, s_n has a convergent subsequence.

6.3 Cauchy sequences

Definition 43. s_n is a Cauchy sequence if for any positive real ϵ, there exist an integer N such that for any $p > N$ and $q > N$, then $|s_p - s_q| < \epsilon$

Theorem 44. A sequence s_n is convergent if and only if s_n is a Cauchy sequence.

7 Chapter 19

Theorem 45. If a series $\sum a_n$ is convergent then $\lim\limits_{n\to\infty} a_n = 0$.

7.1 Series of nonnegative terms

Remark: The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ is divergent if $p \leqslant 1$.

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$.

Theorem 46. Suppose that $u_n \geqslant 0$ for every n, then the series $\sum u_n$ is convergent iff $\displaystyle\sum_{k=1}^{n} u_k$ is bounded.

Theorem 47. (Comparison test) Let $\sum a_n$ and $\sum b_n$ series with non negative terms such that for all the index greater than some N, $0 \leqslant a_n \leqslant b_n$.

- If $\sum b_n$ is convergent, then $\sum a_n$ is convergent.
- If $\sum a_n$ is divergent, then $\sum b_n$ is divergent.

Theorem 48. Let $\sum a_n$ and $\sum b_n$ series with positive terms such that $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = c \neq 0$, then either $\sum a_n$ and $\sum b_n$ are both convergent or $\sum a_n$ and $\sum b_n$ are both divergent.

Theorem 49. **(Ratio Test)** Let $\sum a_n$ and $\sum b_n$ series with positive terms such that $\dfrac{a_{n+1}}{a_n} \leqslant \dfrac{b_{n+1}}{b_n}$ for any n, then

- If $\sum b_n$ is convergent, then $\sum a_n$ is convergent.
- If $\sum a_n$ is divergent, then $\sum b_n$ is divergent.

Theorem 50. Let a_n be a sequence with positive terms.

- If $\lim_{n\to\infty} \dfrac{u_{n+1}}{u_n} = r < 1$ then $\sum u_n$ is convergent.
- If $\lim_{n\to\infty} \dfrac{u_{n+1}}{u_n} = r > 1$ then $\sum u_n$ is divergent.
- If $\lim_{n\to\infty} \dfrac{u_{n+1}}{u_n} = 1$, then the theorm does not give any conclusion.

Theorem 51. **(Comparison to a geometric series)** If for any n, $\dfrac{u_{n+1}}{u_n} < r < 1$, then $\sum u_n$ is convergent.

Theorem 52. **(comparison with an integral)** Given a positive function f that is non increasing on a interval $[1, \infty)$. Then the series $\sum f(n)$ and $\displaystyle\int_1^{\infty} f(t)\mathrm{d}t$ are both convergent or they are both divergent.

7.2 Absolute convergence and conditional convergence

Definition 53. Given a series $\sum u_n$, $\sum u_n$ is absolutely convergent if $\sum |u_n|$ is convergent.

Theorem 54. If $\sum u_n$ is absolutely convergent, then $\sum u_n$ is convergent.

Theorem 55. Given a series $\sum u_n$, if a_n is the subsequence of u_n corresponding to the positive terms.If b_n is the subsequence of u_n corresponding to the non positive terms (possibly completed by 0 terms is one of the sequence is finite)

- If $\sum u_n$ is absolutely convergent, then $\sum a_n$ and $\sum b_n$ are convergent and $\sum u_n = \sum a_n + \sum b_n$.
- If $\sum u_n$ is conditionally convergent, then both $\sum a_n$ and $\sum b_n$ are divergent.

Definition 56. If $\sum u_n$ is convergent and not absolutely convergent, then $\sum u_n$ is conditionally convergent.

Theorem 57. **(Alternating series)** A series $\sum u_n$ such that its terms alternate between positive and negative, and such that $|u_n|$ is decreasing toward 0 is convergent.

8 Chapter 20

8.1 Pointwise convergence

Definition 58. Given a sequence of functions f_n ,
the sequence f_n is pointwise convergent to f on an interval I if for any x in I, $\lim_{n\to\infty} f(x) = f(x)$.
Equivalently

$$\forall x \in I, \forall \epsilon > 0, \exists N_{\epsilon,x} > 0 \text{ such that } \forall n > N, |f_n(x) - f(x)| < \epsilon$$

Theorem 59. (Cauchy pointwise criterion) A sequence of functions f_n is pointwise convergent on an interval I if

$$\forall x \in I, \forall \epsilon > 0, \exists N, \text{ such that } \forall n > N, \forall p > N, |f_n(x) - f_p(x)| < \epsilon$$

8.2 Uniform convergence

Definition 60. A sequence of functions f_n converges uniformly to f on an interval I if

$$\forall \epsilon > 0, \exists N_\epsilon > 0 \text{ such that } \forall n > N \text{ and } \forall x \in I, |f_n(x) - f(x)| < \epsilon$$

Theorem 61. (Cauchy uniform criterion) A sequence of functions f_n is uniformly convergent on an interval I if

$$\forall \epsilon > 0, \exists N > 0 \text{ such that } \forall n > N, \forall p > N, \forall x \in I \, |f_n(x) - f_p(x)| < \epsilon$$

Theorem 62. (Uniform convergence for series of functions) Let $\sum u_n(x)$ be a series of functions defined on an interval I. Let M_n a sequence of upper bounds for $|u_n(x)|$ on I ($M_n \geqslant |u_n(x)|$ for any x in I) , if $\sum M_n$ is convergent,
then $\sum u_n(x)$ converges uniformly on I.

8.3 Results about uniform convergent sequences

Theorem 63. (Continuity of the limit) If a sequence of continuous functions f_n on an interval I is uniformly convergent to f on I, then f is continuous on I.

Theorem 64. (Integral of the limit) If f_n is a sequence of continuous functions on an interval I that is uniformly convergent to f on I, then f is integrable on I and for any a and b in I,

$$\int_a^b f(t)\mathrm{d}t = \lim_{n\to\infty} \int_a^b f_n(t)\mathrm{d}t$$

Theorem 65. (Differentiability of the limit) If f_n is a sequence of differentiable function on an interval I such that

- $f_n(x)$ is convergent to $f(x)$ for any x in I
- f_n' converges uniformly to a function g on I.
- f_n' is continuous on I

then f is differentiable, its derivative is g and $f' = g$ is continuous.